Derivation of the KdV equation from 2D Euler

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1 Preliminaries

I will begin with the Euler equations in the presence of gravity for flat layer of constant density fluid over a flat bottom with infinite horizontal extent and depth, D. If the acceleration of gravity is denoted by g then we non-dimensionalize the spatial variables using D, the velocity using $c = \sqrt{gD}$ (which is the shallow water gravity wave speed) and the time using $\sqrt{D/g}$. The fluid density, ρ_* scales out of the problem and the pressure is non-dimensionalized to the fluid density multiplied by the gravity wave speed squared, $\rho_* gD$.

The momentum equation and incompressibility constraint are

$$\begin{aligned} \rho\left[u_t + \vec{u}\cdot\vec{\nabla}u\right] + p_x &= 0 \\ \rho\left[w_t + \vec{u}\cdot\vec{\nabla}w\right] + p_z &= -\rho \\ \nabla\cdot\vec{u} &= 0. \end{aligned} \tag{1}$$

For a constant density, incompressible fluid, the density equation

$$\rho_t + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \tag{2}$$

has a distributional (weak) solution

$$\rho(x,z,t) = \Theta\left(F(x,z,t)\right) \tag{3}$$

where

$$F(x,z,t) = 0 \tag{4}$$

is function which describes the interface of the fluid. The boundary conditions are

$$p = 0 \quad \text{at} \quad F(x, z, t) = 0 \tag{5}$$

i.e. the pressure vanishes at the top interface and

$$w = 0 \quad \text{at} \quad z = 0 \tag{6}$$

i.e. the vertical velocity vanishes at the bottom boundary. In the asymptotic derivation, we must focus on the case where the upper boundary is a single valued function of x for all t, therefore

$$F(x, z, t) = 1 + h(x, t) - z \tag{7}$$

and $h(x, t)$ is called the height of the interface.

1.1 Kinematic condition for the interface

For a general interface, simply substitute (3) into (2) which yields

$$\delta(F(x, z, t)) \; [F_t + uF_x + wF_z] = 0. \tag{8}$$

This equation implies that F is advected by the velocity (u, w) on the interface

$$F_t + uF_x + wF_z = 0 \quad \text{on} \quad F(x, z, t) = 0, \tag{9}$$

which is called the kinematic condition of the interface. In the case where the interface is a single valued function of x this further reduces to an equation for the height

$$h_t + uh_x - w = 0 \quad \text{on} \quad z = 1 + h(x, t). \tag{10}$$

1.2 Equilibrium

The equilibrium is given by

$$u = w = h = 0 \tag{11}$$

so that the interface is at

$$z = 1 \tag{12}$$

and the equilibrium pressure is

$$p = 1 - z \tag{13}$$

2 Asymptotic Expansion

We seek solutions near the equilibrium, i.e. for small initial data. To this end, we seek an asymptotic expansion and see if we can make it valid for long times. Therefore substitute

$$\begin{bmatrix} u \\ w \\ p \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1-z \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} u_1 \\ w_1 \\ p_1 \\ h_1 \end{bmatrix} + \epsilon^2 \begin{bmatrix} u_2 \\ w_2 \\ p_2 \\ h_2 \end{bmatrix} + \mathcal{O}(\epsilon^3) \tag{14}$$

into equations (1) and (10) using the boundary conditions (5) and (6).

It is important to point out that the pressure boundary condition is evaluated on the moving interface,

$$p(x, 1 + h(x,t), t) = 0. \tag{15}$$

Substituting the pressure boundary condition into the asymptotic expansion we find

$$\begin{aligned} 0 &= p(x, 1 + \epsilon h_1(x,t) + \epsilon^2 h_2(x,t) + \mathcal{O}(\epsilon^2), t) \\ &= 1 - 1 - \epsilon h_1 - \epsilon^2 h_2 + \ldots \\ &\qquad \epsilon p_1(x, 1 + \epsilon h_1 + \epsilon^2 h_2, t) + \epsilon^2 p_2(x, 1 + \epsilon h_1 + \epsilon^2 h_2, t) + \mathcal{O}(\epsilon^3) \\ &= -\epsilon h_1 - \epsilon^2 h_2 + \epsilon p_1(x,1,t) + \epsilon^2 h_1 p_{1,z}(x,1,t) + \epsilon^2 p_2(x,1,t) + \mathcal{O}(\epsilon^3). \end{aligned}$$

where $p_{1,z}$ is the z-partial derivative of p_1. Collecting the terms order by order we find the boundary conditions

$$p_1(x,1,t) = h_1(x,t) \tag{16}$$

and

$$p_2(x,1,t) = h_2(x,t) - h_1(x,t)\, p_{1,z}(x,1,t). \tag{17}$$

Thankfully, the vertical velocity boundary condition at the lower boundary is much easier

$$w_i(x,0,t) = 0 \quad \forall i \tag{18}$$

since it is a fixed boundary.

These kinds of free boundary problems constitute an exceedingly interesting and, as of yet not completely explored, class of problems; KdV notwithstanding.

2.1 $\mathcal{O}(\epsilon)$: Linear Theory

Carefully exploring the linear theory of a non-linear problem is truly the most important step. The structure of the spectrum of the linear operator and the eigenfunctions will tell you almost everything you need to know to make progress in the asymptotic expansion.

The equations are $\mathcal{O}(\epsilon)$ are

$$\begin{aligned} \partial_t u_1 + \partial_x p_1 &= 0 \\ \partial_t w_1 + \partial_z p_1 &= 0 \\ \partial_x u_1 + \partial_z w_1 &= 0 \\ \partial_t h_1 - w_1 &= 0 \end{aligned} \tag{19}$$

where it must be emphasized that the first three equations apply in $0 \leq z \leq 1$ whereas the fourth equation is evaluated at $z = 1$. I have also changed the notation so that the partial derivatives don't get mixed up with the subscripts of the orders. The boundary conditions are

$$p_1(x, 1, t) = h_1(x, t) \quad \text{and} \quad w_1(x, 0, t) = 0 \tag{20}$$

2.1.1 Stationary vortex solutions

There are 2 types of solutions to these equations - this corresponds to the Hodge Weyl decomposition that I mentioned in class. The first class are "vorticity" solutions and have zero vertical velocity at the top boundary. It is straightforward to show that, for these solutions, there is a streamfunction ψ such that

$$\begin{aligned} u_1 &= -\partial_z \psi \\ w_1 &= \partial_x \psi \end{aligned} \tag{21}$$

so that (u_1, w_1) automatically satisfy the incompressibility constraint $\partial_x u_1 + \partial_z w_1 = 0$. By taking the curl of the first two equations in (19), we find

$$\partial_t \left(\partial^2_{xx} \psi + \partial^2_{zz} \psi \right) = 0. \tag{22}$$

Further requiring $w_1(x, 1, t) = 0$ means that

$$\partial_t h_1 = 0. \tag{23}$$

Therefore these vorticity solutions correspond to stationary (time independent) flows. The stream function is

$$\psi = \tilde{\psi}_{n,k} \sin(n\pi z) \cos(k(x - x_0)) \tag{24}$$

for any n integer and any real values of k, x_0 and amplitude $\tilde{\psi}_{n,k}$. Each of these stream functions correspond to a circulation below the surface of the fluid. In an ideal (Euler) fluid, such a circulation is time independent.

These stream function solutions do not generate surface waves (at least at lowest order) and are not the solutions we are interested in. (Notice, also, that these solutions have a significant amplitude at all depths in the fluid - the solutions we are interested are surface waves whose velocity field decays sharply away from the top boundary).

2.1.2 Linear surface waves

Taking the divergence of the first two equations in (19) and applying the divergence-free constraint we find

$$\partial^2_{xx} p_1 + \partial^2_{zz} p_1 = 0 \tag{25}$$

in $0 < z < 1$, $-\infty < x < \infty$. Since the whole problem is a system of linear, constant coefficient PDEs and the domain is unbounded in x, we can Fourier transform in x, or equivalently seek wave-like solutions

$$p_1(x, z, t) = \tilde{p}_1(z; k)\, e^{i(kx - \omega t)} \tag{26}$$

which, upon substituting into (25) and (20) yields the boundary value problem

$$\frac{d^2 \tilde{p}_1}{dz^2} - k^2 \tilde{p}_1 = 0 \tag{27}$$

with the boundary condition to be explained in a moment. The solutions to this equation are exponentials; it is convenient to combine them into hyperbolic sines and cosines, and we write

$$\tilde{p}_1 = \tilde{p}_1^+ \cosh(k(z - 1)) + \tilde{p}_1^- \sinh(k(z - 1)) \tag{28}$$

for any constants, $\tilde{p}_1^+, \tilde{p}_1^-$; since the boundary conditions on p_1 will be applied at $z = 1$, I have written the functions as functions of $z - 1$. Combining with (26) we find

$$p_1(x, z, t) = e^{i(kx - \omega t)} \left[\tilde{p}_1^+ \cosh(k(z - 1)) + \tilde{p}_1^- \sinh(k(z - 1))\right] \tag{29}$$

where the tilde variables depend on k and correspond to the Fourier transform of the initial data.

Fourier transforming the remaining variables, we have

$$
\begin{aligned}
u_1 &= e^{i(kx-\omega t)}\ \tilde{u}_1(z;k) \\
w_1 &= e^{i(kx-\omega t)}\ \tilde{w}_1(z;k) \\
h_1 &= e^{i(kx-\omega t)}\ \tilde{h}_1(k)
\end{aligned}
\tag{30}
$$

which, upon substituting in (19) and (20) we find

$$
\begin{aligned}
-i\omega\tilde{u}_1 + ik\left[\tilde{p}_1^+\cosh(k(z-1)) + \tilde{p}_1^-\sinh(k(z-1))\right] &= 0 \\
-i\omega\tilde{w}_1 + k\left[\tilde{p}_1^+\sinh(k(z-1)) + \tilde{p}_1^-\cosh(k(z-1))\right] &= 0 \\
ik\tilde{u}_1 + \frac{d\tilde{w}_1}{dz} &= 0 \\
-i\omega\tilde{h}_1 - \tilde{w}_1(1;k) &= 0 \\
\tilde{w}_1(0;k) &= 0 \\
\tilde{p}_1^+ &= \tilde{h}_1.
\end{aligned}
\tag{31}
$$

Though the notation can get cumbersome, in principle this is simply a matrix problem for the Fourier coefficients. Solving for $\tilde{u}_1$ from the first equation

$$
\tilde{u}_1 = \frac{k}{\omega}\left[\tilde{p}_1^+\cosh(k(z-1)) + \tilde{p}_1^-\sinh(k(z-1))\right]
\tag{32}
$$

and for $\tilde{w}_1$

$$
\tilde{w}_1 = -i\frac{k}{\omega}\left[\tilde{p}_1^+\sinh(k(z-1)) + \tilde{p}_1^-\cosh(k(z-1))\right].
\tag{33}
$$

Notice that the third equation in (31) is automatically satisfied by these solutions since this equation (the incompressibility constraint) was used to construct the elliptic problem for p_1 in equation (25) from which this solution arose. Now, eliminating $\tilde{h}_1$ from the fourth equation and substituting for $\tilde{w}_1$ in the fourth and fifth equations we find

$$
\begin{aligned}
-i\omega\tilde{p}_1^+ + i\frac{k}{\omega}\tilde{p}_1^- &= 0 \\
-\tilde{p}_1^+\sinh(k) + \tilde{p}_1^-\cosh(k) &= 0
\end{aligned}
\tag{34}
$$

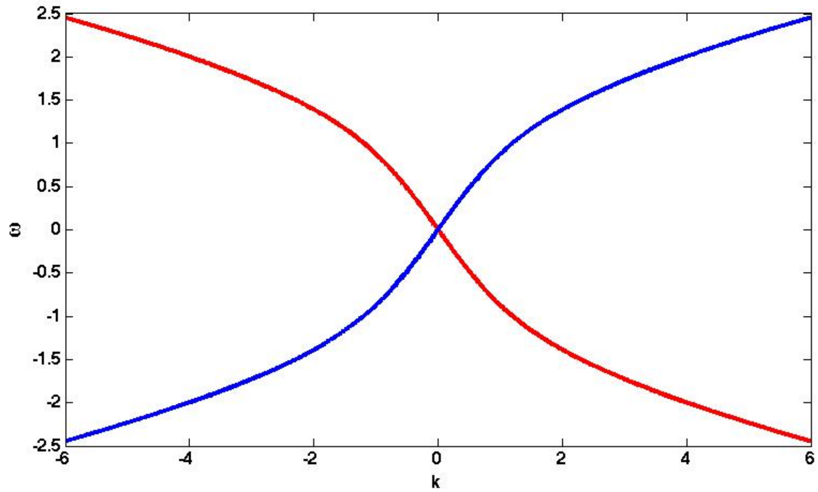

Figure 1: Dispersion relation for linear surface waves. Blue (red) branch corresponds to right (left) going waves. As $k \to 0$

or written as a matrix

$$\begin{bmatrix} -\omega & \frac{k}{\omega} \\ \sinh(k) & -\cosh(k) \end{bmatrix} \begin{bmatrix} \tilde{p}_1^+ \\ \tilde{p}_1^- \end{bmatrix} = 0 \tag{35}$$

Of course, the matrix must be singular for a solution to exist; setting the determinant to zero yields

$$\omega \cosh(k) - \frac{k \sinh(k)}{\omega} = 0.$$

The solution to this equation is called the dispersion relation and relates the frequency of a wave to its wavenumber

$$\omega = \omega(k) = \pm\sqrt{k \tanh(k)}. \tag{36}$$

Notice that there are two solutions, known as "branches" of the dispersion relation. These correspond to rightward and leftward propagating waves, respectively and are plotted in figure 1.

Regarding the eigenvector, we find

$$\tilde{p}_1^- = \tanh(k)\, \tilde{p}_1^+ = \tanh(k)\, \tilde{h}_1. \tag{37}$$

After a bit of algebra (and using the addition formula for hyperbolic sine and cosine), we can show that the components of the linear eigenfunction are

$$\begin{bmatrix} u_1 \\ w_1 \\ p_1 \\ h_1 \end{bmatrix} = \tilde{h}_1\, e^{i(kx-\omega t)} \begin{bmatrix} \pm\sqrt{\frac{k}{\tanh(k)}}\frac{\cosh(kz)}{\cosh(k)} \\ \mp i\sqrt{\frac{k}{\tanh(k)}}\frac{\sinh(kz)}{\cosh(k)} \\ \frac{\cosh(kz)}{\cosh(k)} \\ 1 \end{bmatrix}. \tag{38}$$

A lot of things can be said here. The positive/negative signs correspond to right and left going waves, respectively. I have chosen to scale everything to the actual height of the wave, $\tilde{h}_1$ at each wavenumber. The fields u, p, h are all in phase with one another whereas w (as evidenced by the multiplication by i) is $\pi/2$ out of phase with the others. This means that where the height is maximum, so too is the pressure and horizontal velocity, but the vertical velocity is zero. Conversely, when the height is zero, so too are the pressure and horizontal velocity, but the vertical velocity is maximum.

2.1.3 Shallow and deep water limits

The limit of long waves is the shallow water limit from which will arise KdV. A layer is considered shallow if the wavelength of the wave is longer than the depth of the layer - this is the limit $k \ll 1$. In this limit the dispersion relation becomes

$$\omega \approx \pm k \tag{39}$$

which correspond to non-dispersive waves traveling to the right or the left. Non-dispersive waves are those that we have most experience with, i.e. light and sound at the audible frequencies. The eigenfunction in this limit is

$$\begin{bmatrix} u_1 \\ w_1 \\ p_1 \\ h_1 \end{bmatrix} \approx \tilde{h}_1\, e^{i(kx-\omega t)} \begin{bmatrix} \pm\frac{2-(kz)^2}{2-k^2} \\ \mp i\frac{kz}{2-k^2} \\ \frac{2-(kz)^2}{2-k^2} \\ 1 \end{bmatrix} \approx \tilde{h}_1\, e^{ik(x\mp t)} \begin{bmatrix} \pm 1 \\ \mp ikz \\ 1 \\ 1 \end{bmatrix}. \tag{40}$$

This last result merits a comment or two. In the limit of long waves (or shallow water), the horizontal velocity and pressure are nearly constant as a function of height in the fluid layer. On the other hand, the vertical velocity is a linear function of height in this limit. Moreover, the vertical velocity at the top of the layer is $\mathcal{O}(k)$ (remembering that we are in the $k \ll 1$ limit) despite the fact that all of the other fields are $\mathcal{O}(1)$. This structure of the eigenfunction is what is exploited to create the long wave scaling.

The deep water limit occurs when $k \gg 1$ in which case

$$\omega = \pm\sqrt{k}. \tag{41}$$

Without getting into the details, this result implies that shorter waves travel more slowly than longer waves - something that can be directly observed when staring longingly at the sea while standing on the California bluffs above the Pacific Ocean. With a little algebra you can see that

$$\begin{bmatrix} u_1 \\ w_1 \\ p_1 \\ h_1 \end{bmatrix} \approx \tilde{h}_1\, e^{i(kx-\omega t)} \begin{bmatrix} \pm\sqrt{k}\; e^{k(z-1)} \\ \mp i\sqrt{k}\; e^{k(z-1)} \\ e^{k(z-1)} \\ 1 \end{bmatrix}. \tag{42}$$

In the deep water limit, the velocity fields and pressure perturbation decay exponentially away from the surface of the water. Moreover, for a fixed height of the wave, both components of the velocity are $\mathcal{O}(\sqrt{k})$ stronger than the pressure and height perturbations.

So we conclude, shallow water waves really "feel the bottom" of the basin they travel in. Deep water waves decay exponentially with depth in the basin. The difference between shallow and deep water waves is not due to their amplitude (since this is linear theory, it cannot be about their amplitude) but rather it is due to their wavelength; shallow (deep) water waves have wavelengths larger (less) than the depth of the basin.

3 Long Wave Scaling

Consider a function $f(x)$ which is $o(1)$ everywhere in x (i.e. $\max_{\mathrm{x}}|f(x)| \leq 1$). Consider the Fourier transform

$$f(x) = \int_{\mathbf{R}} \tilde{f}(k) e^{ikx}\, dk \tag{43}$$

and the derivative

$$\frac{d}{dx} f(x) = \int_{\mathbf{R}} ik\; \tilde{f}(k) e^{ikx}\, dk \tag{44}$$

Now let us suppose that $\tilde{f}$ has support over long wavelengths, i.e. small k. Using the Fourier transform of the derivative, it is straightforward to estimate

$$\max \frac{d}{dx} f(x) < [\max k] \; [\max_x f(x)] \,. \tag{45}$$

Now this is a little sloppy and there are caveats everywhere, but the point is, if the function is supported on small wavenumbers, then the derivative is much smaller than the function.

This motivates a long wave scaling as follows. We introduce a scaled variable, X

$$x = \frac{X}{\delta} \tag{46}$$

where $\delta \ll 1$. We will imagine that all functions $f(x)$ can be written $f(X/\delta)$ therefore

$$\frac{\partial}{\partial x} f = \delta \frac{\partial}{\partial X} f \quad \text{where} \quad \frac{\partial}{\partial X} f \sim \mathcal{O}(1). \tag{47}$$

This is equivalent to supposing that the Fourier transform of f is supported at small wavenumbers.

Notice that in the long wave limit $k \sim \delta$ and from the dispersion relation at long wavelengths, equation (39), we see that

$$\omega \sim \delta. \tag{48}$$

This suggests a long time scaling for the time derivatives

$$t = \frac{T}{\delta} \tag{49}$$

consequently the time derivatives become

$$\frac{\partial}{\partial t} f = \delta \frac{\partial}{\partial T} f \quad \text{where} \quad \frac{\partial}{\partial T} f \sim \mathcal{O}(1). \tag{50}$$

Lastly we look at the eigenfunction in equation (40). We see that, for $\mathcal{O}(1)$ perturbations of the height (i.e. $\tilde{h}_1 \sim \mathcal{O}(1)$ the eigenvector has $\mathcal{O}(1)$ pertubations of the horizontal velocity and the pressure. However, the maximum vertical velocity, which occurs at $z = 1$ is $\mathcal{O}(k) \sim \delta \ll 1$. This motivates the scaling of the variables

$$\begin{bmatrix} u \\ w \\ p \\ h \end{bmatrix} \rightarrow \begin{bmatrix} u \\ \delta w \\ p \\ h \end{bmatrix}. \tag{51}$$

Substituting the new variables into equations (1) and (10) and canceling some powers of δ yields

$$\begin{aligned} \frac{\partial u}{\partial T} + u\frac{\partial u}{\partial X} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial X} &= 0 \\ \delta^2\left[\frac{\partial w}{\partial T} + u\frac{\partial w}{\partial X} + w\frac{\partial w}{\partial z}\right] + \frac{\partial p}{\partial z} &= -1 \\ \frac{\partial u}{\partial X} + \frac{\partial w}{\partial z} &= 0 \\ \frac{\partial h}{\partial T} + u\frac{\partial h}{\partial X} - w &= 0 \end{aligned} \tag{52}$$

with boundary conditions

$$\begin{aligned} w &= 0 \quad \text{on } \ z = 0 \\ p &= 0 \quad \text{on } \ z = 1 + h(X,T). \end{aligned} \tag{53}$$

and keeping in mind that the third equation in (52) is to be evaluated only at the upper surface. Together (52) and (53) constitute the long wave scaled surface water equations.

As a last step, I introduce the weakly non-linear assumption. Instead of doing the full asymptotic expansion of the previous section, I will only introduce the perturbation; that is to say, I will seek solutions which are near the equilibrium

$$\begin{aligned} u &\to \epsilon u \\ w &\to \epsilon w \\ p &\to 1 - z + \epsilon p \\ h &\to \epsilon h \end{aligned} \tag{54}$$

where I am simply replacing the original variables by their ϵ rescaled counterparts. Technically this is the assumption we make on the initial data; i.e. that initial perturbations have amplitude ϵ. However, the idea is to ensure that the asymptotic expansion maintains this assumption for a "long time". Substituting (54) into the long wave equations (52) and their boundary conditions (53) yields the weakly nonlinear long wave equations

$$\begin{aligned}
\frac{\partial u}{\partial T}+\frac{\partial p}{\partial X} &= -\epsilon\left[u\frac{\partial u}{\partial X}+w\frac{\partial u}{\partial z}\right]\\
\frac{\partial p}{\partial z} &= -\delta^2\left[\frac{\partial w}{\partial T}\right]-\epsilon\delta^2\left[u\frac{\partial w}{\partial X}+w\frac{\partial w}{\partial z}\right]\\
\frac{\partial u}{\partial X}+\frac{\partial w}{\partial z} &= 0\\
\frac{\partial h}{\partial T}-w &= -\epsilon\left[u\frac{\partial h}{\partial X}\right]
\end{aligned}\tag{55}$$

with boundary conditions

$$\begin{aligned}
w=0 &\quad \text{on} \;\; z=0\\
p=h &\quad \text{on} \;\; z=1+\epsilon h(X,T)
\end{aligned}\tag{56}$$

where, again, the height equation is to be only evaluated at the upper surface.

3.1 Linear theory of the long wave scaling

In the limit $\delta\to 0$ and $\epsilon\to 0$ the linear long wave equations are

$$\begin{aligned}
\frac{\partial u}{\partial T}+\frac{\partial p}{\partial X} &= 0\\
\frac{\partial p}{\partial z} &= 0\\
\frac{\partial u}{\partial X}+\frac{\partial w}{\partial z} &= 0\\
\frac{\partial h}{\partial T}-w &= 0
\end{aligned}\tag{57}$$

with boundary conditions

$$\begin{aligned}
w=0 &\quad \text{on} \;\; z=0\\
p=h &\quad \text{on} \;\; z=1.
\end{aligned}\tag{58}$$

3.1.1 Linear energy

This is an energy conserving system and we can define the (linear) energy as follows. Multiply the first equation in (57) by u, the second by w, add them together and integrate

$$
\begin{aligned}
0 &= \int_0^1 \int_{-\infty}^{\infty} \left[u\,\frac{\partial u}{\partial T} + u\,\frac{\partial p}{\partial X} + w\frac{\partial p}{\partial z} \right]\, dxdz \\
&= \int_0^1 \int_{-\infty}^{\infty} \left[\frac{1}{2}\frac{\partial \left(u^2\right)}{\partial T} + \frac{\partial \left(up\right)}{\partial X} + \frac{\partial \left(wp\right)}{\partial z} \right]\, dXdz \quad \text{using incompressibility} \\
&= \frac{d}{dT}\int_0^1 \int_{-\infty}^{\infty} \frac{u^2}{2}\, dXdz + \int_0^1 (u\,p)\Big|_{X\to-\infty}^{X\to\infty} dz + \int_{-\infty}^{\infty} (wp)\Big|_{z=0}^{z=1} dX
\end{aligned}
$$

using Green's theorem. Now, assuming that the solutions are either periodic in x or that they decay rapidly enough (i.e. are $\mathbf{L}^2$) then we can neglect the second integral in the above expression. The third integral is simplified by using the boundary conditions at the bottom ($w = 0$) and top ($p = h$) of the layer. The expression simplifies to

$$
0 = \frac{d}{dT}\int_0^1 \int_{-\infty}^{\infty} \frac{u^2}{2}\, dXdz + \int_{-\infty}^{\infty} w(x,1,t)h(x,t)\, dX. \tag{59}
$$

Now add to this expression h times the third equation in (57) integrated over x. After the dust settles, the wh terms cancel and we are left with the conservation of energy, i.e.

$$
E \equiv \int_{-\infty}^{\infty} \left\{ \left[\int_0^1 \frac{u^2}{2}\, dz \right] + h^2 \right\}\, dX = \text{constant}. \tag{60}
$$

3.1.2 Solution of the linear long wave equations

I now solve the linear long wave dynamics in equation (57). Notice that, since p is not a function of z from the second equation in (57), then nor can u be a function of z. Applying the upper boundary condition on the pressure, we find

$$
p = h \quad \text{for all } z \tag{61}
$$

where, again, the height equation is to be only evaluated at the upper surface.

Therefore, we can integrate the incompressibility constraint with respect to z and applying the lower boundary condition we find

$$
w = -z\,\frac{\partial u}{\partial X}. \tag{62}
$$

Upon eliminating w from the fourth equation and p from the first equation we find

$$\begin{aligned} \frac{\partial u}{\partial T} + \frac{\partial h}{\partial X} &= 0 \\ \frac{\partial h}{\partial T} + \frac{\partial u}{\partial X} &= 0. \end{aligned} \tag{63}$$

Subtracting the X derivative of the first equation from the T derivative of the second equation, we find the wave equation

$$\frac{\partial^2 h}{\partial T^2} - \frac{\partial^2 h}{\partial X^2} = 0 \tag{64}$$

for which we have the D'Alembert solution

$$h(x,t) = \frac{h_0(X+T) + h_0(X-T)}{2} + \frac{1}{2}\int_{X-T}^{X+T} v_0(s)\, ds \tag{65}$$

where $v_0(X)$ is the initial velocity of the interface, in this example

$$v_0(X) = \left.\frac{\partial h}{\partial T}\right|_{T=0} = -\frac{\partial u_0}{\partial X}. \tag{66}$$

where $u_0(X)$ is the initial horizontal velocity. Therefore, we can perform the integral in (65) to find

$$h(x,t) = \frac{[h_0(x+T) + h_0(X-T)] - [u_0(X+T) - u_0(X-T)]}{2} \tag{67}$$

Another way to construct the solution to (63) is to define the Riemann invariants,

$$\begin{aligned} R &= \frac{u+h}{2} \\ L &= \frac{u-h}{2} \end{aligned} \tag{68}$$

and, upon adding and subtracting the equations in (63) we find

$$\begin{aligned} \frac{\partial R}{\partial T} + \frac{\partial R}{\partial X} &= 0 \\ \frac{\partial L}{\partial T} - \frac{\partial L}{\partial X} &= 0 \end{aligned} \tag{69}$$

which are two, decoupled, transport equations for the left going wave, L, and the right going wave, R. The solutions are

$$\begin{aligned} R(X,t) = R_0(X-T) &= \frac{u_0(X-T)+h_0(X-T)}{2} \\ L(X,T) = L_0(X+T) &= \frac{u_0(X+T)-h_0(X+T)}{2} \end{aligned} \tag{70}$$

for any initial functions, $h_0(X), u_0(X)$. Subtract R and L in (70) then dividing by 2 to yield the solution for the height of the wave in (67). To get the horizontal velocity, add R and L and divide by two.

3.1.3 Solvability condition for a forced transport equation

Consider the rightward transport equation forced by a specified forcing

$$\frac{\partial R}{\partial T} + \frac{\partial R}{\partial X} = F(X,T). \tag{71}$$

Defining the characteristics

$$\xi = X - T, \quad \eta = X + T \tag{72}$$

we find

$$\begin{aligned} \frac{\partial}{\partial \xi} &= -\frac{\partial}{\partial T} + \frac{\partial}{\partial X} \\ \frac{\partial}{\partial \eta} &= \frac{\partial}{\partial T} + \frac{\partial}{\partial X} \end{aligned} \tag{73}$$

so that the forced transport equation (71) becomes

$$\frac{\partial R}{\partial \eta} = F\left(\frac{\eta+\xi}{2}, \frac{\eta-\xi}{2}\right) \tag{74}$$

Therefore the transport equations have bounded solutions if

$$\left| \int_{-\infty}^{\infty} F\left(\frac{\eta+\xi}{2}, \frac{\eta-\xi}{2}\right)\, d\eta \right| < \infty \quad \forall\, \xi. \tag{75}$$

That is to say, solutions are well behaved if the integral along each characteristic remains finite.

Another way to think of this is to consider the linear equation

$$\frac{\partial R}{\partial \eta} = G(\xi, \eta) \tag{76}$$

multiply by an (as of yet) arbitrary function $R^\dagger(\xi, \eta)$ and integrate over all ξ, η

$$\int\int_{\mathbf{R}^2} R^\dagger \frac{\partial R}{\partial \eta}\, d\xi d\eta = \int\int_{\mathbf{R}^2} R^\dagger G(\xi, \eta)\, d\xi d\eta \tag{77}$$

Integrating this expression by parts we find the adjoint problem

$$\int_{\mathbf{R}} R^\dagger R \bigg|_{\eta\to-\infty}^{\eta\to\infty} d\xi - \int\int_{\mathbf{R}^2} R \frac{\partial R^\dagger}{\partial \eta} d\xi d\eta = \int\int_{\mathbf{R}^2} R^\dagger G(\xi, \eta)\, d\xi d\eta. \tag{78}$$

Using compact initial data for R means

$$\lim_{\eta\to-\infty} R(\xi, \eta) = 0. \tag{79}$$

And if we define the adjoint linear operator

$$-\frac{\partial R^\dagger}{\partial \eta} = 0 \tag{80}$$

then the integral simplifies to

$$\int_{\mathbf{R}} R^\dagger R \bigg|^{\eta\to\infty} d\xi = \int\int_{\mathbf{R}^2} R^\dagger G(\xi, \eta)\, d\xi d\eta. \tag{81}$$

so that solutions, $R(\xi, \eta)$ remain bounded if the integral on the right hand side of (81) remains bounded for all solutions $R^\dagger(\xi, \eta)$ of (80). Notice that the adjoint eigenfunction (the solution of of (80)) is

$$R^\dagger(\xi, \eta) = \Psi(\xi) \tag{82}$$

for any well behaved function $\Psi(\xi)$. Substituting this solution into (81) yields

$$\int_{\mathbf{R}} \Psi(\xi) R \bigg|^{\eta\to\infty} d\xi = \int\int_{\mathbf{R}^2} \Psi(\xi) G(\xi, \eta)\, d\xi d\eta. \tag{83}$$

for all well behaved (i.e. compactly supported) functions $\Psi(\xi)$. In particular, we usually think of the δ-function as appropriate test functions, whereby

$$\Psi(\xi) = \delta(\xi - \xi_*) \quad \forall \quad \xi,\ \xi_* \tag{84}$$

and using these test functions in (83) the ξ integral simplifies to yield

$$\lim_{\eta \to \infty} R(\xi_*, \eta) = \int_{-\infty}^{\infty} G(\xi_*, \eta)\, d\eta \tag{85}$$

Therefore solutions $R(\xi, \eta)$ are well behaved as $\eta \to \infty$ if

$$\left| \int_{-\infty}^{\infty} G(\xi, \eta)\, d\eta \right| < \infty \quad \forall \quad \xi \tag{86}$$

which is the same condition as (75).

Of course an equivalent solvability condition applies to the left-going waves.

3.1.4 Solvability condition for the linear long wave equations

In order to construct the adjoint of the long wave linear operator, take the system of equations (57) - but include inhomogeneous terms on the right hand side of each equation as we did for the transport equation above; F_u, F_w, F_p, F_h denote the "forcings" on each of the equations in (57). Now multiply the first by $u^\dagger(X, z, T)$, the second by $w^\dagger(X, z, T)$ the third by $h^\dagger(X, T)$ and the fourth by $p^\dagger(X, z, T)$. The functions with daggers will turn out to be the adjoint eigenfunctions. Add the resulting equations and integrate over the whole domain and all time to find

$$\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u^\dagger \frac{\partial u}{\partial T} + u^\dagger \frac{\partial p}{\partial X} + w^\dagger \frac{\partial p}{\partial z} + h^\dagger \frac{\partial h}{\partial T} - h^\dagger w + p^\dagger \frac{\partial u}{\partial X} + p^\dagger \frac{\partial w}{\partial z} \right\} dT dX dz$$
$$= \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u^\dagger F_u + w^\dagger F_w + p^\dagger F_p + h^\dagger F_h \right\} dT dX dz$$

where I have used the fact that the integral over z on the h equation is equal to one in order to write the expression most concisely. Notice that this is the same exercise used to construct the energy and when the daggered quantities are equal to their non-daggered counterparts, this expression would reproduce conservation of energy in the absence of the inhomogeneous terms.

Now integrate the right hand side of this expression by parts

$$
\begin{aligned}
&\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -u\frac{\partial u^\dagger}{\partial T} - p\frac{\partial u^\dagger}{\partial X} - p\frac{\partial w^\dagger}{\partial z} - h\frac{\partial h^\dagger}{\partial T} - u\frac{\partial p^\dagger}{\partial X} - w\frac{\partial p^\dagger}{\partial z} \right\} \, dTdXdz \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -w(1)h^\dagger + w^\dagger(1)\, p(1) - w^\dagger(0)\, p(0) + p^\dagger(1)\, w(1) - p^\dagger(0)\, w(0) \right\} \, dTdX \\
&+ \int_0^1 \int_{-\infty}^{\infty} \left[u^\dagger u + h^\dagger h \right] \Big|_{T\to-\infty}^{T\to\infty} \, dXdz + \int_0^1 \int_{-\infty}^{\infty} \left[u^\dagger p + p^\dagger u \right] \Big|_{X\to-\infty}^{X\to\infty} \, dTdz \\
= &\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u^\dagger F_u + w^\dagger F_w + p^\dagger F_p + h^\dagger F_h \right\} \, dTdXdz
\end{aligned}
$$

where the function evaluations in the second line indicate the z values. Notice that for compact initial data, u, p the last integral in the third line vanishes. The idea is to determine the conditions for which the second last integral in the third line also remains finite; let us denote this integral with a $\mathcal{I}$ for "increase", since it corresponds to the increase of the solution after the interaction of waves.

Rearranging the terms and applying the lower boundary condition on w and upper boundary condition on p the expression for the increase becomes

$$
\begin{aligned}
&\mathcal{I} - \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u\left[\frac{\partial u^\dagger}{\partial T} + \frac{\partial p^\dagger}{\partial X}\right] + w\left[\frac{\partial p^\dagger}{\partial z}\right] + p\left[\frac{\partial u^\dagger}{\partial X} + \frac{\partial w^\dagger}{\partial z}\right] \right\} \, dTdXdz \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -h\left[\frac{\partial h^\dagger}{\partial T} - w^\dagger(1)\right] - w(1)\left[h^\dagger - p^\dagger(1)\right] - p(0)\left[w^\dagger(0)\right] \right\} \, dTdX \\
= &\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u^\dagger F_u + w^\dagger F_w + p^\dagger F_p + h^\dagger F_h \right\} \, dTdXdz.
\end{aligned}
\tag{87}
$$

This demonstrates the construction of the adjoint to the linear operator in (57); by setting the terms in square parentheses independently to zero, we have the adjoint linear operator (and its boundary conditions)

$$
\begin{aligned}
\frac{\partial u^\dagger}{\partial T} + \frac{\partial p^\dagger}{\partial X} &= 0 \\
\frac{\partial p^\dagger}{\partial z} &= 0 \\
\frac{\partial u^\dagger}{\partial X} + \frac{\partial w^\dagger}{\partial z} &= 0 \\
\frac{\partial h^\dagger}{\partial T} - w^\dagger(1) &= 0 \\
w^\dagger(0) &= 0 \\
h^\dagger - p^\dagger(1) &= 0
\end{aligned}
\tag{88}
$$

and have shown that, if the daggered quantities are solutions of (88), then the increase integral is

$$
\mathcal{I} = \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u^\dagger F_u + w^\dagger F_w + p^\dagger F_p + h^\dagger F_h \right\} \, dT dX dz \tag{89}
$$

It is clear that the equations and boundary conditions for the adjoint of the linear operator (88) are identical to those of the original linear operator (57). In fact, they appear in the integral in (87) with a negative sign. Such a linear operator is said to be *skew self-adjoint*, meaning the adjoint is the negative of the original linear operator. In such examples, the eigenvalues are purely imaginary - and in this case correspond to neutrally stable waves. The beauty of a skew self-adjoint operator is that if one solves the original linear operator, then that eigenfunction is also an eigenfunction of the adjoint.

Finally, we seek solutions whose increase (after the interaction of the waves) remains finite, therefore

$$
\begin{gathered}
\mathcal{I} < \infty \quad \Longrightarrow \\
\left| \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ u^\dagger F_u + w^\dagger F_w + p^\dagger F_p + h^\dagger F_h \right\} \, dT dX dz \right| < \infty
\end{gathered}
\tag{90}
$$

when the daggered quantities are solutions of (88). This is the solvability condition of the linear operator in the presence of inhomogeneous terms.

3.1.5 Explicit, real valued solution of the linear operator and its adjoint

Piecing together everything we have developed the solution of the long wave equations is

$$\begin{bmatrix} u \\ w \\ p \\ h \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -h_0(X+T)+h_0(X-T)+u_0(X+T)+u_0(X-T) \\ -z\left[-h_0'(X+T)+h_0'(X-T)+u_0'(X+T)+u_0'(X-T)\right] \\ h_0(X+T)+h_0(X-T)-u_0(X+T)+u_0(X-T) \\ h_0(X+T)+h_0(X-T)-u_0(X+T)+u_0(X-T) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} u_0(X+T)-h_0(X+T) \\ -z\left[u_0'(X+T)-h_0'(X+T)\right] \\ -\left[u_0(X+T)-h_0(X+T)\right] \\ -\left[u_0(X+T)-h_0(X+T)\right] \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_0(X-T)+h_0(X-T) \\ -z\left[u_0'(X-T)+h_0'(X-T)\right] \\ u_0(X-T)+h_0(X-T) \\ u_0(X-T)+h_0(X-T) \end{bmatrix} \tag{91}$$

where primes denote derivatives with respect to the argument. Substituting R_0 and L_0

$$\begin{bmatrix} u \\ w \\ p \\ h \end{bmatrix} = \begin{bmatrix} L_0(X+T) \\ -zL_0'(X+T) \\ -L_0(X+T) \\ -L_0(X+T) \end{bmatrix} + \begin{bmatrix} R_0(X-T) \\ -zR_0'(X-T) \\ R_0(X-T) \\ R_0(X-T) \end{bmatrix}. \tag{92}$$

We conclude that for any two initial functions $R_0(X)$, $L_0(X)$, or equivalently $h_0(X)$, $u_0(X)$ the solution is given by (92), or equivalently, (91).

Just as the solution to the linear problem consists of a superposition of two functions given in (92), so two there are two eigenfunctions of the adjoint problem. For any functions $\Psi_+(X)$ and $\Psi_-(X)$ the eigenfunctions of the adjoint are

$$\begin{bmatrix} \Psi_+(X-T) \\ -z\Psi_+'(X-T) \\ \Psi_+(X-T) \\ \Psi_+(X-T) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Psi_-(X+T) \\ -z\Psi_-'(X+T,) \\ -\Psi_-(X+T) \\ -\Psi_-(X+T) \end{bmatrix}. \tag{93}$$

3.2 Multiple scales theory of the weakly nonlinear long wave equations

Anticipating that an asymptotic expansion of the solution will break down at second order because of resonant interactions from the nonlinear terms, we pose a multiple time scale expansion as follows. Introduce a long time scale

$$\tau = \epsilon T \tag{94}$$

so that all of the derivatives with respect to time in (55) become

$$\frac{\partial}{\partial T} \longrightarrow \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau}. \tag{95}$$

Substituting (95) intoe (55) yields the multiple time scale, weakly nonlinear long wave equations

$$\begin{aligned}
\frac{\partial u}{\partial T} + \frac{\partial p}{\partial X} &= -\epsilon \left[\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial X} + w \frac{\partial u}{\partial z} \right] \\
\frac{\partial p}{\partial z} &= -\delta^2 \left[\frac{\partial w}{\partial T} \right] - \epsilon \delta^2 \left[\frac{\partial w}{\partial \tau} + u \frac{\partial w}{\partial X} + w \frac{\partial w}{\partial z} \right] \\
\frac{\partial u}{\partial X} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial h}{\partial T} - w &= -\epsilon \left[\frac{\partial h}{\partial \tau} + u \frac{\partial h}{\partial X} \right]
\end{aligned} \tag{96}$$

which come with the boundary conditions

$$\begin{aligned}
w &= 0 \quad \text{on} \;\; z = 0 \\
p &= h \quad \text{on} \;\; z = 1 + \epsilon h(X, T).
\end{aligned} \tag{97}$$

Notice that I have switched the order of the height and divergence equation in (96) so that it matches the order of the adjoint eigenfunction in (92).

3.2.1 Distinguished limit

There are two small parameters in this problem; δ measures the ratio of the depth of the layer to typical length scales in the horizontal while ϵ measures the ratio of the initial horizontal velocity to the shallow water wave speed. The lowest order terms on the right hand side of (96) are either ϵ or δ^2. Since we expect (and it will be the case) that the solvability condition will be applied at first order in the perturbation theory and we would like to create a theory in which linear terms balance the non-linear terms (called "dispersion balanced nonlinearity"), we seek a distinguished limit where

$$\delta^2 \sim \epsilon \quad \text{specifically} \quad \delta^2 = \epsilon. \tag{98}$$

3.2.2 Asymptotic expansion

We now pose a regular asymptotic expansion for the variables

$$\begin{bmatrix} u \\ w \\ p \\ h \end{bmatrix} = \begin{bmatrix} u_0 \\ w_0 \\ p_0 \\ h_0 \end{bmatrix} + \epsilon \begin{bmatrix} u_1 \\ w_1 \\ p_1 \\ h_1 \end{bmatrix} + \mathcal{O}(\epsilon^3) \tag{99}$$

which differs from the expression (14) because of the redefinition of the variables in the weakly non-linear long wave equations (55).

3.2.3 $\mathcal{O}(\epsilon^0)$:

At this order, we have the linear long wave theory all over again, except now there is no restriction on the behavior of the wave with respect to the long time variable, τ. The solution is given by (92) with the additional flexibility that the functions L_0 and R_0 can depend on τ also

$$\begin{bmatrix} u_0 \\ w_0 \\ p_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} L_0(X+T,\tau) \\ -zL_0'(X+T,\tau) \\ -L_0(X+T,\tau) \\ -L_0(X+T,\tau) \end{bmatrix} + \begin{bmatrix} R_0(X-T,\tau) \\ -zR_0'(X-T,\tau) \\ R_0(X-T,\tau) \\ R_0(X-T,\tau) \end{bmatrix}. \tag{100}$$

We focus on compactly supported initial data, which means that both L_0 and R_0 are $\mathbf{L}^2$ functions in X.

3.2.4 $\mathcal{O}(\epsilon^1)$:

Notice that $\epsilon\delta^2 = \epsilon^2$ so that the nonlinear term on the right hand side of the vertical velocity equation is higher order. The first order equations are

$$\begin{aligned} \frac{\partial u_1}{\partial T} + \frac{\partial p_1}{\partial X} &= -\left[\frac{\partial u_0}{\partial \tau} + u_0\frac{\partial u_0}{\partial X} + w_0\frac{\partial u_0}{\partial z}\right] \\ \frac{\partial p_1}{\partial z} &= -\left[\frac{\partial w_0}{\partial T}\right] \\ \frac{\partial u_1}{\partial X} + \frac{\partial w_1}{\partial z} &= 0 \\ \frac{\partial h_1}{\partial T} - w_1 &= -\left[\frac{\partial h_0}{\partial \tau} + u_0\frac{\partial h_0}{\partial X}\right] \end{aligned} \tag{101}$$

with boundary conditions

$$\begin{array}{ll} w_1 = 0 & \text{at} \quad z = 0 \\ p_1 = h_1 - h_0 \frac{\partial p_0}{\partial z} & \text{at} \quad z = 1. \end{array} \tag{102}$$

But p_0 is independent of z, so the last boundary condition simplifies to

$$p_1 = h_1 \quad \text{at} \quad z = 1. \tag{103}$$

So the linear operator at first order is exactly the linear operator at lowest order. The first order equations are forced by the lower order wave (on the right hand side). Therefore, to solve this problem, we require that the projection of the right hand side of (101) on either adjoint eigenfunction in (93) for any functions $\Psi_\pm(X)$ must equal zero.

There are five types of terms on the right hand side of (101):

1. Linear functions of $R_0(X-T)$
2. Linear functions of $L_0(X+T)$
3. Quadratic nonlinear functions of $R_0(X-T)$
4. Quadratic nonlinear functions of $L_0(X+T)$
5. Quadratic nonlinear functions that look like $L_0(X+T)\,R_0'(X-T)$ or $L_0'(X+T)\,R_0(X-T)$.

What these mean, physically, is that the first order problem is forced by the rightward wave (1 and 3), the leftward wave (2 and 4) and the interaction of the rightward and leftward wave (5 above). When we project against the adjoint eigenfunction we will need to consider integrals of the form

$$\int\int_{\mathbf{R}^2} \Psi_\pm(X \mp T) F(X,T)\; dT dX \tag{104}$$

where $F(X,T)$ represents on of the five types of terms listed above.

For the terms which fall under 1 and 3 above, the forcing by the rightward wave, $F(X,T) = f(X-T)$ and the integrals become

$$\int\int_{\mathbf{R}^2} \Psi_\pm(X \mp T) f(X-T)\; dT dX. \tag{105}$$

Defining the right and leftward characteristics

$$\xi = X - T, \quad \eta = X + T \tag{106}$$

so that

$$dT\, dX = \frac{1}{2} d\xi\, d\eta. \tag{107}$$

The two types of integrals in (105) become

$$\int\int_{\mathbf{R}^2} \Psi_+(\xi) f(\xi)\; d\xi d\eta \quad \text{and} \quad \int\int_{\mathbf{R}^2} \Psi_-(\eta) f(\xi)\; d\xi d\eta \tag{108}$$

For compactly supported initial data, $f(X)$ is integrable. Moreover, we consider compactly supported, integrable functions, $\Psi_\pm(X)$ (such as the δ-function). Therefore the second integral in (108) is

$$\left|\int\int_{\mathbf{R}^2} \Psi_-(\eta) f(\xi)\; d\xi d\eta\right| = \left|\left[\int_{\mathbf{R}} \Psi_-(\eta)\; d\eta\right]\right| \; \left|\left[\int_{\mathbf{R}} f(\xi)\; d\xi\right]\right| < \infty \tag{109}$$

since both of the functions Ψ_-, f are integrable.

However, the first integral is insidious since

$$\int\int_{\mathbf{R}^2} \Psi_+(\xi) f(\xi)\; d\xi d\eta = \left|\left[\lim_{a\to\infty} \int_{-a}^{a} d\eta\right]\right| \; \left|\left[\int_{\mathbf{R}} \Psi_+(\xi) f(\xi)\; d\xi.\right]\right| \tag{110}$$

Now, the second integral in this expression exists; for example if we let $\Psi(\xi) = \delta(\xi - \xi_*)$ then it evaluates to $f(\xi_*)$. **However, the first integral does not exist!** Therefore, in order that the "increase" remain bounded we require

$$f(\xi) = 0 \quad \forall \xi \tag{111}$$

A similar argument can be made for the terms of the form 2 and 4 from the right hand side of (101). If we denote these terms by $g(X + T) = g(\eta)$ then the integrals become

$$\int\int_{\mathbf{R}^2} \Psi_+(\xi) g(\eta)\; d\xi d\eta \quad \text{and} \quad \int\int_{\mathbf{R}^2} \Psi_-(\eta) g(\eta)\; d\xi d\eta \tag{112}$$

and now **the second integral does not exist unless**

$$g(\eta) = 0 \quad \forall \eta. \tag{113}$$

Finally, let us denote the terms of the form 5 as $\gamma(X+T, X-T) = \gamma(\xi, \eta)$ and consider the integrals

$$\int\int_{\mathbf{R}^2} \Psi_+(\xi)\gamma(\xi,\eta)\, d\eta d\xi \quad \text{and} \quad \int\int_{\mathbf{R}^2} \Psi_-(\eta)\gamma(\xi,\eta)\, d\xi d\eta. \tag{114}$$

Since the integral of γ with respect to either of its arguments exists and is finite, then these integrals simplify to

$$\int\int_{\mathbf{R}} \Psi_+(\xi)\alpha(\xi)\, d\xi \quad \text{and} \quad \int\int_{\mathbf{R}} \Psi_-(\eta)\beta(\eta)\, d\eta \tag{115}$$

and since $\Psi_\pm$ are both compactly supported, then each of these integrals is finite.

We conclude that there are only two solvability conditions for (101)

- The projection of the righthand side of (101) against arbitrary rightward moving wave profiles is zero, i.e. $f(\xi) = 0$ for all ξ.
- The projection of the righthand side of (101) against arbitrary leftward moving wave profiles is zero, i.e. $g(\eta) = 0$ for all η.

3.2.5 The Korteweg-deVries Equation

Each of the solvability conditions will give a KdV equation for either the right moving or the left moving wave. I will focus on the right moving wave as the argument (as we have seen above) is the same for the left moving wave. Substitute the expression for the right going wave from equation (100) into left hand side of equation (101) and project on the adjoint eigenfunction Ψ_+ from equation (93)

$$\begin{aligned} -\int_0^1 \int\int_{\mathbf{R}^2} \Bigg\{ &\Psi_+(\xi)\left[\frac{\partial R_0(\xi,\tau)}{\partial \tau} + R_0(\xi,\tau)\frac{\partial R_0(\xi,\tau)}{\partial \xi}\right] \ldots \\ &- z\frac{d\Psi_+(\xi)}{d\xi}\left[z\frac{\partial^2 R_0(\xi,\tau)}{\partial \xi^2}\right] \ldots \\ &+\Psi_+(\xi)\left[\frac{\partial R_0(\xi,\tau)}{\partial \tau} + R_0(\xi,\tau)\frac{\partial R_0(\xi,\tau)}{\partial \xi}\right]\Bigg\}\, d\xi d\eta dz \quad < \quad \infty \end{aligned} \tag{116}$$

notice that I have split the lines to emphasize from which line in the equations the terms arose. Also, I have used the fact that, for the lowest order eigenfunction

$$\frac{\partial u_0}{\partial z} = 0 \tag{117}$$

and

$$\frac{\partial w_0}{\partial T} = zR_0''(\xi,\tau) = z\frac{\partial^2 R_0(\xi,\tau)}{\partial \xi^2} \tag{118}$$

to simplify the expression. Note again that primes denote derivatives with respect to the first argument, which is ξ; i.e. the variable describing the characteristic. Collecting terms, cancelling the leading negative sign and performing the z-integration, (116) becomes

$$\begin{aligned} \int\int_{\mathbf{R}^2} \Bigg\{ 2\Psi_+(\xi)\left[\frac{\partial R_0(\xi,\tau)}{\partial \tau} + R_0(\xi,\tau)\frac{\partial R_0(\xi,\tau)}{\partial \xi}\right] \ldots \\ -\frac{1}{3}\frac{d\Psi_+(\xi)}{d\xi}\left[\frac{\partial^2 R_0(\xi,\tau)}{\partial \xi^2}\right]\Bigg\}\, d\xi d\eta \quad < \quad \infty \end{aligned} \tag{119}$$

Finally, integrate the last term by parts, factor $\Psi_+(\xi)$ in front of all of the terms and divide by two. Again, the boundary terms at $|\xi| \to \infty$ vanish for compactly supported data and the expression can finally be written as

$$\int\int_{\mathbf{R}^2} \Psi_+(\xi)\left\{\frac{\partial R_0(\xi,\tau)}{\partial \tau} + R_0(\xi,\tau)\frac{\partial R_0(\xi,\tau)}{\partial \xi} + \frac{1}{6}\frac{\partial^3 R_0(\xi,\tau)}{\partial \xi^3}\right\} d\xi d\eta < \infty \tag{120}$$

for all compactly supported $\Psi_+(\xi)$. Again, we can use $\Psi_+(\xi) = \delta(\xi - \xi_*)$ or it is straightforward enough to see that, in order that the integral remain finite, the expression in curly parentheses must vanish

$$\frac{\partial R_0}{\partial \tau} + R_0\frac{\partial R_0}{\partial \xi} + \frac{1}{6}\frac{\partial^3 R_0}{\partial \xi^3} = 0 \tag{121}$$

where R_0 is a function of ξ, τ. **This is the Korteweg-deVries equation.**

Just to simplify notation a step further, we note that

$$R_0(\xi,\tau) = R_0(X - T, \tau) = h_0(X, T, \tau) \equiv H(X, T, \tau) \tag{122}$$

that is to say that the right going wave amplitude is exactly equal to the height of the rightward traveling wave (after it has passed all of the leftward traveling wave). Therefore, the KdV equation is an expression for the modulation of the height of a rightward traveling surface wave in a frame of reference moving at the linear wave speed, $\xi = X - T$.