The forced wave equation

J.A. Biello

November 28, 2017

In this note, I would like to flesh out D'Alembert's principle for the forced wave equation on the line.

Consider the forced wave equation

$$u_{tt} - c^2 u_{xx} = h(x,t), \quad \text{in} \quad t \geq 0, x \in \mathbf{R} \tag{1}$$

for a height $u(x,t)$ and a forcing $h(x,t)$ with initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x). \tag{2}$$

Introduce the characteristic coordinates

$$\begin{aligned} \xi &= x - ct \\ \eta &= x + ct \end{aligned} \tag{3}$$

with the inverse transformation

$$\begin{aligned} x &= \frac{\eta + \xi}{2} \\ t &= \frac{\eta - \xi}{2c}. \end{aligned} \tag{4}$$

Since we are interesting in $t \geq 0$ then we require $\eta \geq \xi$. Writing the height in the characteristic coordinates

$$U(\xi,\eta) \equiv u(x(\xi,\eta), t(\xi,\eta) \quad \text{equivalently} \quad u(x,t) = U(\xi(x,t), \eta(x,t)). \tag{5}$$

Computing the derivatives in the new coordinates, we find

$$\begin{aligned} u_t &= U_\xi \xi_t + U_\eta \eta_t = -cU_\xi + cU_\eta \\ u_{tt} &= -c\left[U_{\xi\xi}\xi_t + U_{\xi\eta}\eta_t\right] + c\left[U_{\eta\xi}\xi_t + U_{\eta\eta}\eta_t\right] \\ &= c^2\left[U_{\xi\xi} + U_{\eta\eta}\right] - 2c^2 U_{\xi\eta} \\ u_x &= U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta \\ u_{xx} &= U_{\xi\xi} + U_{\eta\eta} + 2U_{\xi\eta}. \end{aligned} \tag{6}$$

Also, if we let

$$H(\xi,\eta) = h(x(\xi,\eta), t(\xi,\eta)) \tag{7}$$

the wave equation becomes

$$-4c^2 U_{\xi\eta} = H(\xi, \eta). \tag{8}$$

and the initial conditions turn into side conditions. At $t = 0$, $\xi = \eta$. We find

$$U(s,s) = u_0(s) \tag{9}$$

and

$$cU_\eta(s,s) - cU_\xi(s,s) = v_0(s). \tag{10}$$

Taking the s derivative of (9) yields

$$U_\xi(s,s) + U_\eta(s,s) = \frac{du_0(s)}{ds}. \tag{11}$$

Now we can solve for $U_\eta(s,s)$ and $U_\xi(s,s)$, which are the derivatives along characteristics when evaluated along the curve $\xi = s, \ \eta = s$. Equations (10) and (11) imply

$$U_\eta(s,s) = \frac{1}{2}\left[u_0'(s) + \frac{v_0(s)}{c}\right], \quad U_\xi(s,s) = \frac{1}{2}\left[u_0'(s) - \frac{v_0(s)}{c}\right], \tag{12}$$

and the primes are derivatives with respect to the argument.

The next step is to integrate the PDE (8) with respect to ξ. We find

$$U_\eta(\xi,\eta) = F(\eta) - \frac{1}{4c^2}\int_0^\xi H(\xi',\eta)\, d\xi', \tag{13}$$

where the constant of integration can be any function of η. We can also integrate (8) with respect to η to find

$$U_\xi(\xi,\eta) = G(\xi) - \frac{1}{4c^2}\int_0^\eta H(\xi,\eta')\, d\eta', \tag{14}$$

where the lower bound can be any function of ξ.

Now we evaluate (13), (14) along the line $\xi = s, \ \eta = s$ and solve for F and G, respectively to find

$$\begin{aligned} F(s) &= \frac{u_0'(s)}{2} + \frac{v_0(s)}{2c} + \frac{1}{4c^2}\int_0^s H(\xi',s)\, d\xi' \\ G(s) &= \frac{u_0'(s)}{2} - \frac{v_0(s)}{2c} + \frac{1}{4c^2}\int_0^s H(s,\eta')\, d\eta' \end{aligned} \tag{15}$$

Next we integrate (13) after having substituted for $F(\eta)$ from (15)

$$\begin{aligned} U(\xi,\eta) &= g(\xi) + \int_0^\eta \left[\frac{u_0'(s)}{2} + \frac{v_0(s)}{2c} + \frac{1}{4c^2}\int_0^s H(\xi',s)\, d\xi'\right] ds - \frac{1}{4c^2}\int_0^\eta \int_0^\xi H(\xi',s)\, d\xi' ds \\ &\text{[where g}(\xi)\text{ is the ``constant'' of integration]} \\ &= g(\xi) + \frac{u_0(\eta) - u_0(0)}{2} + \frac{1}{2c}\int_0^\eta v_0(s)\, ds - \frac{1}{4c^2}\int_0^\eta \int_s^\xi H(r,s)\, dr ds \end{aligned} \tag{16}$$

where we have used the fundamental theorem of calculus to perform the first integral, and we've changed the lower bounds of integration in the last integral. We could have use the expression for U_ξ, but the solution would have been this same.

The final step is to use the initial data to evaluate g. Evaluate at $\xi = a, \eta = a$ and use $U(a,a) = u_0(a)$. Solving for g we find

$$g(a) = \frac{u_0(a) + u_0(0)}{2} - \frac{1}{2c}\int_0^a v_0(s)ds + \frac{1}{4c^2}\int_0^a \int_s^a H(r,s)\, drds. \tag{17}$$

Substituing (17) into (16) yields

$$\begin{aligned} U(\xi,\eta) &= \frac{u_0(\xi) + u_0(\eta)}{2} + \frac{1}{2c}\int_\xi^\eta v_0(s)\, ds - \frac{1}{4c^2}\int_\xi^\eta \int_s^\xi H(r,s) drds \\ &= \frac{u_0(\xi) + u_0(\eta)}{2} + \frac{1}{2c}\int_\xi^\eta v_0(s)\, ds + \frac{1}{4c^2}\int_\xi^\eta \int_\xi^{\eta'} H(\xi',\eta') d\xi' d\eta' \end{aligned} \tag{18}$$

The last step is to recast the independent variables in terms of x, t. This is straightforward for the terms in u_0, v_0. It is the forcing term that is more subtle. The double integral is an integral over the right angle triangle in the ξ', η' plane with hypotenuse $\xi' = \eta'$, vertical side $\xi' = \xi$ and horizontal side $\eta' = \eta > \xi$. The area element transforms as

$$\begin{aligned} d\xi'\, d\eta' &= \left| \frac{\partial \xi'}{\partial x'}\frac{\partial \eta'}{\partial t'} - \frac{\partial \eta'}{\partial x'}\frac{\partial \xi'}{\partial t'} \right| \, dx'dt' \\ &= |c - (-c)| \; dx'dt' \\ &= 2c\; dx'dt'. \end{aligned} \tag{19}$$

This boundary $\xi' = \eta'$ is equivalent to $t' = 0$. The other two boundaries delimit the boundaries

$$x' - ct' = x - ct, \qquad \text{and} \qquad x' + ct' = x + ct. \tag{20}$$

These are two characteristic lines in the (x', t') plane which cross at the point $t' = t$, $x' = x$. Also they cross the $t' = 0$ line at

$$x' = x - ct, \quad \text{and} \quad x' = x + ct \tag{21}$$

respectively. Therefore the solution in (x, t) coordinates is

$$u(x,t) = \frac{u_0(x+ct) + u_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\, ds + \frac{1}{2c}\int_0^t \int_{x-c(t-t')}^{x+c(t-t')} h(x',t')\, dx'\, dt'. \tag{22}$$

The interpretation of this expression is as follows. Defining the causal cone of a point (x,t) as all the points in the triangle with vertices $(x,t), (x-ct, 0), (x+ct, 0)$, then the solution at (x,t) is given by

1. Half the initial height at each of the other vertices
2. Half the integral (divided by c) of the initial velocity integrated between the other vertices.
3. Half the integral (divided by c) of the acceleration over the causal cone.