The Kepler problem

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The granddaddy of all problems in dynamical systems is the so-called Kepler problem. Isaac Newton invented the calculus in order to solve the equations he had discovered while studying Kepler's laws of planetary motion around a central body under the influence of gravity - the sun. This also goes by the name of the two-body problem and, although it constitutes a moderately high dimensional dynamical system (12 dimensions), the symmetries and conservation laws allow an enormous simplification of the problem so that it can be solved analytically.

We discuss it at the beginning of a course in dynamical systems for many reasons - but most especially because of the beauty of the solution and the techniques that are used to arrive at it.

The three laws are:

1. Planets travel in ellipses around the center of mass of the planet/sun system. In fact, more generally (and not known to Kepler), the possible trajectories in a two body system are any of the conic sections, and nothing else.

2. (the period of the orbit)^2 $\propto$ (the length of the semimajor axis)^3

3. A portion of the planetary orbit can be seen to sweep out a curved wedge whose boundary is a portion of the ellipse and the lines which connect the arc to the focus. Two such regions are swept out with the same area and so the planet traverses those arc in the same amount of time.

Then, of course, there are Newton's three laws of motion:

1. A particle will travel in a straight line at a constant velocity, unless acted upon by an external, unbalanced force.

2. The rate of change of momentum of the particle equals the net force acting on the particle. The momentum is the mass multiplied by the velocity of the particle and so

$$m\frac{d}{dt}\vec{v} = \vec{F} \tag{1}$$

3. The force exerted by a first particle on a second particle is equal to the force exerted by the second particle on the first particle, but in the opposite direction.

1 The equations governing the motion of two bodies.

Consider the positions of a group of particles $\{\vec{x}_i(t), i = 1...N : \mathbf{R} \to \mathbf{R}^{3N}\}$. $\vec{x}_i(t)$ is the trajectory of of the "ith" particle in the group. The velocity of each particle is given by

$$\vec{v}_i = \frac{d\vec{x}_i}{dt}.$$

Therefore Newton's second law can be written

$$m_i \frac{d}{dt}\vec{v}_i = \vec{F}(\vec{x}_j, \vec{v}_j) \tag{2}$$

for each i and for all j.

Let us consider two point masses, m_1, m_2 with locations $\vec{x}_1(t), \vec{x}_2(t)$ acting on one another only through gravity. Newton's law of Gravity staes that the force of the first mass on the second is given by

$$\vec{F}_{1\to 2} = -\frac{Gm_1m_2}{|\vec{x}_1 - \vec{x}_2|^2}\,\hat{r}_{12} = -\frac{Gm_1m_2}{|\vec{x}_1 - \vec{x}_2|^3}\,(\vec{x}_2 - \vec{x}_1) \tag{3}$$

i.e. $\hat{r}_{12}$ is the unit vector in the direction of $(\vec{x}_2 - \vec{x}_1)$ and G is a constant of nature.

Using Newton's second and third laws and the law of gravity we can write the two body problem as an initial value problem consisting of the 12th order system of differential equations

$$\begin{aligned} m_1 \frac{d^2}{dt^2}\vec{x}_1 &= -\frac{Gm_1m_2}{|\vec{x}_1 - \vec{x}_2|^2}\,\hat{r}_{21} \\ m_2 \frac{d^2}{dt^2}\vec{x}_2 &= -\frac{Gm_1m_2}{|\vec{x}_1 - \vec{x}_2|^2}\,\hat{r}_{12} \end{aligned} \tag{4}$$

plus twelve initial conditions (the initial velocities and locations of each of the two particles).

However, this systems is also a Hamiltonian dynamical system and an autonomous central force problem. This means that energy, angular momentum and the center of mass of the system, which we will define below, are all conserved. Such conserved quantities allow us to reduce this system to a second order system which can be solved analytically.

First, let us define the center of mass of a pair of particles

$$\vec{x}_C = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}. \tag{5}$$

Adding the two equations in (4) we find

$$\frac{d^2}{dt^2}\vec{x}_c = 0 \tag{6}$$

which means that the center of mass of the two body system does not accelerate. This, in fact, is a result of first law because all of the forces acting on the two bodies are internal to the system of the two bodies - this system cannot force itself, it must be forced from outside. Since we are not considering outside forces, then the system as a whole (i.e. its center of mass) cannot accelerate.

Equation (5) is a second order ODE for $\vec{x}_c(t)$, to solve it, just integrate twice

$$\vec{x}_c(t) = \vec{v}_0\, t + \vec{x}_0 \tag{7}$$

meaning that the center of mass of the system moves in a line through space which is determined by the initial conditions. Notice that the real solar system is actually acted on by the gravitational force from the rest of the galaxy, so is in orbit around the galaxy - but we're not getting into all of that just yet.

The solution strategy we follow involves a series of transformations which either have mathematical expediency or physical relevance. We are going to describe the system of two bodies not using $\vec{x}_1, \vec{x}_2$ but rather using $\vec{x}_C$, the center of mass and the mass weighted deviation from the center of mass, $\vec{x}$, defined by

$$\begin{aligned} \vec{x}_1 &= \vec{x}_c + \frac{m_2}{m_1 + m_2}\vec{x} \\ \vec{x}_2 &= \vec{x}_c - \frac{m_1}{m_1 + m_2}\vec{x}. \end{aligned} \tag{8}$$

Notice that the separation of the particles is simply

$$\vec{x}_1 - \vec{x}_2 = \vec{x}. \tag{9}$$

Dividing each equation in (4) by the masses on the left hand sides, and then taking the difference, we find

$$\frac{d^2}{dt^2}\vec{x} = -\frac{G(m_1 + m_2)}{|\vec{x}|^2}\hat{x} \tag{10}$$

Notice that this is three second order ODEs for each of the components of $\vec{x}$ and it does not depend on $\vec{x}_c$. Of course, the initial conditions can be expressed in terms of these new variables, also. Therefore, in the center of mass/separation variables, the twelfth order dynamical system breaks up into two, not coupled, sixth order dynamical systems, (6) and (10), the first of which, we have already solved. We now proceed to solve the second one.

2 Conservation of Energy

The gravitational force in (3) can be written as the gradient of a potential. The custom in physics is to write it as the negaitve gradient of the gravitational potential energy, V

$$\vec{F} = -\nabla V. \tag{11}$$

Specifically, we can see that if the potential energy is

$$V = -\frac{G m_1 m_2}{|\vec{x}_1 - \vec{x}_2|} \tag{12}$$

then

$$\vec{F}_{2\to 1} = -\nabla_1 V \quad \text{and} \quad \vec{F}_{1\to 2} = -\nabla_2 V \tag{13}$$

where ∇_1 is the gradient with respect to $\vec{x}_1$ and ∇_2 is the gradient with respect to $\vec{x}_2$. Since V is a function of $(\vec{x}_1 - \vec{x}_2)$, then the fact that $\vec{F}_{2\to 1} = -\vec{F}_{1\to 2}$ follows.

The space $(\vec{x}_1, \vec{x}_2) \in \mathbf{R}^6$ is often denoted the *configuration space* in physics, this is not to be confused with the *phase space* $(\vec{x}_1, \vec{x}_2, \frac{d}{dt}\vec{x}_1, \frac{d}{dt}\vec{x}_2) \in \mathbf{R}^{12}$ - more on phase space later.

As an exercise to the reader (a simple application of the chain rule) show that the time derivative of the potential energy along a trajectory in configuration space is given by the dot product of the velocity with the gradient (taken in configuration space), i.e.

$$\frac{d}{dt}V(\vec{x}_1(t), \vec{x}_2(t)) = \frac{d\vec{x}_1}{dt} \cdot \nabla_1 V + \frac{d\vec{x}_2}{dt} \cdot \nabla_2 V. \tag{14}$$

Now we take the dot product of each equation in (4) with its respective velocity, add the two equations, use (14), and re-arrange to find

$$\begin{aligned} m_1 \frac{d}{dt}\vec{x}_1 \cdot \frac{d^2}{dt^2}\vec{x}_1 + m_2 \frac{d}{dt}\vec{x}_2 \cdot \frac{d^2}{dt^2}\vec{x}_2 + \frac{d}{dt}V = 0 \\ m_1 \frac{d}{dt}\frac{1}{2}\left|\frac{d\vec{x}_1}{dt}\right|^2 + m_2 \frac{d}{dt}\frac{1}{2}\left|\frac{d\vec{x}_2}{dt}\right|^2 + \frac{d}{dt}V = 0 \\ \frac{d}{dt}\left[\frac{m_1|\vec{v}_1|^2 + m_2|\vec{v}_2|^2}{2} + V\right] = 0 \end{aligned}$$

which means that the term in parentheses is contant along trajectories. This is *energy* of the system, which is conserved

$$E(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2) \equiv \frac{m_1|\vec{v}_1|^2 + m_2|\vec{v}_2|^2}{2} - \frac{Gm_1 m_2}{|\vec{x}_1 - \vec{x}_2|} \tag{15}$$

and

$$\frac{dE}{dt} = 0 \quad \text{along trajectories.} \tag{16}$$

It is the sum of kinetic energy (the velocities) and potential energy.

We can write the energy in term of the center of mass and separation and their respective velocities

$$\begin{aligned} E(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2) &= \mathcal{E}(\vec{x}_c, \vec{v}_c, \vec{x}, \vec{v}) \\ &= \frac{(m_1 + m_2)}{2}|\vec{v}_c|^2 + \frac{m_1 m_2}{2(m_1 + m_2)}|\vec{v}|^2 - \frac{Gm_1 m_2}{|\vec{x}|} \end{aligned} \tag{17}$$

where

$$\vec{v} = \frac{d}{dt}\vec{x}. \tag{18}$$

Notice, it too is the sum of the kinetic energy and potential energy - but now the kinetic energy is split up into the kinetic energy of the center of mass of the system (associated with

$\vec{v}_c$) and kinetic energy of the relative separation of the particles (associated with $\vec{v}$). Note that the total mass of the system is $m_T = m_1 + m_2$ and the quantity

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{19}$$

is often called *the reduced mass.*

If, instead of constructing the energy from the dynamical system expressed in terms of the original variables (equations 4) we constructed the energy using the separated system describing the center of mass (equation 6) and the relative separation (equation 10) we would find that not only is the total energy conserved but so too is the kinetic energy of the center of mass

$$E_C(\vec{v}_c) = \frac{(m_1 + m_2)}{2} |\vec{v}_c|^2 \tag{20}$$

and the total internal energy of the system

$$\begin{aligned} E_I(\vec{x}, \vec{v}) &= \frac{m_1 \, m_2}{2(m_1 + m_2)} |\vec{v}|^2 - \frac{G m_1 m_2}{|\vec{x}|} \\ &= \mu \left[\frac{|\vec{v}|^2}{2} - \frac{G(m_1 + m_2)}{|\vec{x}|} \right] \end{aligned} \tag{21}$$

Show that each of these energies is conserved by the two body problem. The fact that these two energies are separately conserved is a consequence of the fact that the center of mass of the system does not couple to the relative separation of the two bodies. The energy in (21) is conserved by the dynamics of (10).

3 Conservation of angular momentum

Lets re-write (10) by multplying by μ

$$\mu \frac{d^2}{dt^2} \vec{x} = -\frac{G m_1 m_2}{|\vec{x}|^2} \hat{x}. \tag{22}$$

Taking the cross product of this equation with $\vec{x}$, the right hand side is zero since $\hat{x} \times \vec{x} = 0$. The equation becomes

$$\begin{aligned} \left[\mu \frac{d^2}{dt^2} \vec{x} \right] \times \vec{x} \ &= 0 \\ \frac{d}{dt} \left[\mu \vec{v} \times \vec{x} \right] &= 0 \end{aligned}$$

which means the vector in parentheses is conserved; it is called the *angular momentum*

$$\vec{M}(t) \equiv \mu \vec{x} \times \vec{v} = \text{constant}. \tag{23}$$

The fact that a vector is conserved by the dynamics affords an enormous simplification. The trajectory $\vec{x}(t)$ is a curve in $\mathbf{R}^3$ and $\vec{v}(t)$ is the tangent vector to the curve. $\vec{M}$ is perpendicular to both (since it is the cross product) and since it points in the same direction for all time, this means that the dynamics lie in the fixed (in time) plane which is perpendicular to $\vec{M}$. So the dynamics are actually in $\mathbf{R}^2$.

4 Polar Coordinates

At this stage one usually introduces spherical polar coordinates to solve the problem. However, due to conservation of angular momentum, we already know that the dynamics lie along a plane and we can make this plane the equator of spherical coordinates, or better yet introduce polar coordinates (r, θ, z) whereby

$$\begin{aligned}
\vec{x} &= r\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) + z\hat{k} = r\hat{r} + z\hat{z} \\
\vec{v} &= \dot{r}\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) + r\dot{\theta}\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) + \dot{z}\hat{k} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{k} \\
\frac{d}{dt}\vec{v} &= \ddot{r}\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) + \dot{r}\dot{\theta}\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) \\
&\quad + \frac{d}{dt}\left[r\dot{\theta}\right]\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) - r(\dot{\theta})^2\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) + \ddot{z}\hat{k} \\
&= \left[\ddot{r} - r\dot{\theta}^2\right]\hat{r} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\theta} + \ddot{z}\hat{k}
\end{aligned} \tag{24}$$

where dots denote time derivatives and $\hat{r}, \hat{\theta}, \hat{k}$ are the unit vectors in cylindrical polar coordinates. The angular momentum is

$$\begin{aligned}
\vec{M} &= \mu\,\vec{x} \times \vec{v} \\
&= \mu\left[-rz\dot{\theta}\hat{r} + (\dot{r}z - \dot{z}r)\,\hat{\theta} + r^2\dot{\theta}\hat{k}\right]
\end{aligned} \tag{25}$$

As $\vec{x}$ changes in time, so do the unit vectors $\hat{r}, \hat{\theta}$. Again, since we will rotate the initial conditions so that $z = \dot{z} = 0$ for all time that the dynamics start and remain in the equatorial plane and

$$\vec{M} = \mu r^2\dot{\theta}\hat{k} \tag{26}$$

which is (consistently) always perpendicular to the equatorial plane. That $\vec{M}$ is constant means that its magnitude (the *specific angular momentum*)

$$\frac{|\vec{M}|}{\mu} = r^2\dot{\theta} \equiv J = \text{constant}. \tag{27}$$

Substitute (24) into (22) and use the fact that $z = \dot{z} = 0$ to find

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= -\frac{G(m_1 + m_2)}{r^2} \\
2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0
\end{aligned} \tag{28}$$

but the second equation is completely redundant since it simply expresses the conservation of angular momentum from equation (27).

5 The dynamics reduced to a second order system.

Now eliminate $\dot{\theta}$ from equation J from the first equation in (28) to find

$$\ddot{r} = \frac{J^2}{r^3} - \frac{Gm_T}{r^2}. \tag{29}$$

This is a second order non-linear ODE for $r(t) > 0$ which must be supplied with initial conditions $r(0), \dot{r}(0)$ and $\dot{\theta}(0)$ - the latter determines the constant J through

$$J = r^2\dot{\theta}. \tag{30}$$

We can also express the energy from (21) in polar coordinates

$$E_I = \mu\left[\frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \frac{Gm_T}{r}\right] \tag{31}$$

or, upon eliminating $\dot{\theta}$

$$E_I = \mu\left[\frac{\dot{r}^2}{2} + \frac{J^2}{2r^2} - \frac{Gm_T}{r}\right]. \tag{32}$$

The energy in (32) is conserved by the dynamics defined by (29).

In physics parlance, the potential energy only depends on the separation of the two objects, r. The total energy is conserved and, since the potential is negative definite but the kinetic energy is positive definite, the objects could, in principle, get arbitrarily close together. This is equivalent to the fact that gravity is an attractive force.

However the energy associated with the angular motion of the particle provides an effective repulsive force in (29) which is manifested as a positive potential in (32). The last two term in the definition of E_I are often called the *effective potential*

$$\mathcal{V}_{eff}(r) \equiv \frac{J^2}{2r^2} - \frac{Gm_t}{r} \tag{33}$$

Clearly (29) can be written

$$\ddot{r} = -\frac{d\mathcal{V}_{eff}}{dr} \tag{34}$$

the effective force being the negative derivative of the effective potential. Therefore we could have derived the same conservation of energy equation (modulo a factor of μ) simply by multiplying 29 by $\dot{r}$ and integrating with respect to time.

If we plot the effective potential, we see three qualitatively different cases. For non-negative total energy and $J \neq 0$, $r_{min} < r < \infty$; that is to say, the orbit can go off to infinity, but cannot come within a certain minimum separation. These will turn out to be hyperbolic orbits.

In general for negative energy, $r_{min} < r < r_{max}$; the orbit remains in some range of distances and these will turn out to be elliptical orbits. Finally, there exists a minimum energy. When the total energy equals this minimum energy, the kinetic energy must be zero and $\dot{r} = 0$; these are clearly circular orbits.

Define r_0 as the location of this minimum, and therefore the radius of the circular orbit for a given angular momentum

$$r_0 = \frac{J^2}{Gm_T}. \tag{35}$$

Then we can use (27) to define the amount of time it takes to traverse one radian of a circular orbit

$$t_0 = \frac{r_0^2}{J} = \frac{J^3}{(Gm_T)^2}. \tag{36}$$

Now let us express r and t in units of r_0 and t_0, respectively. This is called non-dimensionalization, which we will study in detail in 207C, but for now

$$\begin{aligned} r &= r_0\, R \\ t &= t_0\, T \end{aligned} \tag{37}$$

and, upon substituting into (29) we find

$$\frac{r_0}{t_0^2}\frac{d^2R}{dT^2} = \frac{J^2}{r_0^3 R^3} - \frac{Gm_T}{r_0^2 R^2}$$

which simplifies to

$$\frac{d^2R}{dT^2} = \frac{1}{R^3} - \frac{1}{R^2} \tag{38}$$

and the energy becomes

$$E_I = E_* \left[\frac{1}{2}\left(\frac{dR}{dT}\right)^2 + \frac{1}{2R^2} - \frac{1}{R} \right]. \tag{39}$$

where E_* is a combination of the physical constants, G, m_T, μ and the angular momentum, J. In these variables, angular momentum conservation becomes

$$R^2\,\frac{d\theta}{dT} = 1. \tag{40}$$

Again, this is just a cleaner way of expressing the dynamics. The non-dimensional form contains all of the structure of the problem; the structure of the potential energy and force and the conservation laws.

6 Two last transformations which complete the simplification

We have exploited all of the structure of the Kepler problem to affect the most reduction possible if we want to study the orbits in time, $R(T)$. However, an further simplification occurs if, instead, we are willing to study the orbits in space, $R(T(\theta))$. Consider the new variable

$$\kappa(\theta(T)) = \frac{1}{R(T)} \tag{41}$$

then

$$\frac{dR}{dT} = -\frac{1}{\kappa^2}\frac{d\kappa}{d\theta}\frac{d\theta}{dT} = -\frac{d\kappa}{d\theta} \tag{42}$$

where we have used the conservation law to eliminate $\frac{d\theta}{dT}$. Furthermore

$$\frac{d^2R}{dT^2} = -\frac{d^2\kappa}{d\theta^2}\frac{d\theta}{dT} = -\kappa^2\frac{d^2\kappa}{d\theta^2}. \tag{43}$$

Substituting this into the dynamical equation and rearranging yields

$$\frac{d^2\kappa}{d\theta^2} + \kappa = 1 \tag{44}$$

while the energy becomes

$$E_I = E_* \left[\frac{1}{2}\left(\frac{d\kappa}{d\theta}\right)^2 + \frac{(\kappa-1)^2}{2} - 1\right]. \tag{45}$$

Equation (44) is the equation of a forced simple harmonic oscillator. It is extremely straightforward to verify that the solution is

$$\kappa(\theta) = 1 - \alpha\sin(\theta - \theta_0) \tag{46}$$

for arbitrary constants α and θ_0 and $\kappa(\theta) > 0$. Since we're only interested in the shape of this orbit we can set $\theta_0 = 0$ and we find the radius is

$$R(T(\theta)) = \frac{1}{1 - a\sin(\theta)}. \tag{47}$$

Notice that

$$\frac{dR}{d\theta}(\theta = 0) = -a \tag{48}$$

If $|a| > 1$ then there are values of θ for which $R \longrightarrow \infty$. Rearranging (47) we find

$$\begin{aligned} R - \alpha R\sin(\theta) &= 1 \\ R^2 &= 1 + 2aR\sin(\theta) + a^2R^2\sin^2(\theta) \\ X^2 + Y^2 &= 1 + 2aY + a^2Y^2 \end{aligned}$$

where we return to Cartesian coordinates in the last line. If $a = 1$ this is the equation for the parabola

$$Y = \frac{1 - X^2}{2}. \tag{49}$$

If $\alpha = 0$ it is a circle

$$X^2 + Y^2 = 1. \tag{50}$$

Completing the square we find

$$(1 - a^2)^2 \left[Y - \frac{a}{1 - a^2} \right]^2 + (1 - a^2)X^2 = 1. \tag{51}$$

For $|a| < 1$ this is an ellipse aligned along the Y-axis with semi-major axis length

$$A = \frac{1}{1 - a^2} \tag{52}$$

and semi-minor axis length

$$B = \frac{1}{\sqrt{1 - a^2}} \tag{53}$$

and distance from center to foci

$$C = \sqrt{A^2 - B^2} = \frac{a}{1 - a^2} \tag{54}$$

and the focus is located at the origin. The parameter a is also called the eccentricity of the ellipse.

Note, if $|a| > 1$ these are hyperbolae, but a remains the eccentricity.

7 Kepler's Laws

Clearly we have shown the first law, that two bodies interacting under gravity have trajectories which are the conic sections: ellipses or circles for "bound orbits" (those which do not have $r \longrightarrow \infty$.

To show the second law we need only look at the definitions of t_0, which is proportional to the period of the orbit and r_0 which is proportional to the semi-major axis length. From their definitions in (35) and (36) we can eliminate J to find

$$r_0^3 = (Gm_T)t_0^2 \tag{55}$$

which is the second law.

The third law, equal areas in equal times is simply a consequence of the conservation of angular momentum. The area swept out by the orbit in a time Δt is

$$\text{Area} = \Delta t |\vec{x} \times \vec{v}| \tag{56}$$

but the quantity in parentheses is constant throughout the orbit. So at any point in the orbit over equal amounts of time, Δt, the trajectory sweeps out equal areas.

8 Solving for $\theta(T)$

Substituting $R(\theta)$ from (47) into (40) we find that θ can be solved by a simple quadrature

$$\int_0^\theta \frac{d\theta}{(1 - a\sin(\theta))^2} = T. \tag{57}$$

But it can also be thought of graphically

$$\frac{d\theta}{dT} = (1 - a\sin(\theta))^2. \tag{58}$$