To compute the vector potential, both divergence and Stokes' theorem are needed

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1 Stokes' Theorem and the Divergence Theorem

1.1 Stokes' Theorem

This section will be obvious to Hafez, but I state it clearly here to make sure we have the same notation in our minds. I will work with vector fields in $\mathbf{R}^3$. All that is needed of them is that it they be differentiable in some open (simply connected) subset of $\mathbf{R}^3$ at this point in the argument. Stoke's theorem states that for any such vector field

$$\int_C \vec{f} \cdot \hat{T}\, dl = \int\int_\Sigma \left(\nabla \times \vec{f}\right) \cdot \hat{n}\, d\sigma \tag{1}$$

where $d\sigma$ is the area element on the open surface Σ, and $\hat{n}$ is a unit normal to that surface. The boundary of the open surface is the curve C, the line element is dl, and the unit tangent vector is $\hat{T}$. Stokes' theorem works for all surfaces which share the same boundary curve: this is a crucial fact which we will use later to construct the vector potential.

Of course, for Stokes' theorem to work, one needs to have the correct normal and tangent vectors. The construction of $\hat{n}$ and $\hat{T}$ goes as follows. Evaluate the normal vector to the surface, Σ on the boundary curve, C. The correct orientation of normal and tangent is attained if

$$\hat{n} \times \hat{T} = \hat{\mathrm{In}}. \tag{2}$$

It is clear that the right hand side must be a unit vector and it must be perpendicular to $\hat{n}$, therefore it must either point into the surface or away from the surface at the curve. The correct relative orientation is attained if it points into the surface and, therefore, $\hat{\mathrm{In}}$ is the unit vector pointing from the curve *into* the surface.

1.2 Divergence Theorem

The divergence theorem states

$$\int\int_\Sigma \vec{g} \cdot \hat{n}\, d\sigma = \int\int\int_\Omega \nabla \cdot \vec{g}\, dV \tag{3}$$

where Ω is a bounded, simply connected domain in $\mathbf{R}^3$, Σ is the closed surface which forms the boundary of Ω, $d\sigma$ is the area element on that surface, $\hat{n}$ is the outward normal to that surface, and dV is the volume element in the domain.

2 Construction of integral curves and a tube

In this section I construct a "tube" (I *don't* want to call it a flux tube because nothing I say has to do with physics here - but it is related to flux tubes) in the following way. Let us consider a parametrized closed curve in this space, Γ_0, with parameter a whose location is given in parameterized form by $\vec{X}_0(a)$. Let there be a vector field in this space, $\vec{v}(\vec{x})$. Consider a family of curves parameterized by a and t such that

$$\frac{d}{dt}\vec{X}(a,t) = \vec{v}(\vec{X}(a,t)), \tag{4}$$

with initial data on the original parameterized curve

$$\vec{X}(a,0) = \vec{X}_0(a). \tag{5}$$

At every value of "a", the curve $\vec{X}(a,t)$ is the integral curve of the vector field $\vec{v}(\vec{x})$. The collection of these for all different values of "a" create this tube which starts on the curve $\vec{X}_0(a)$. Now consider two functions $T_1(a)$ and $T_2(a)$ such that $T_2(a) \geq T_1(a)$ and define the curves

$$\begin{aligned} \Gamma_1 :& \quad \vec{X}_1(a) = \vec{X}(a, T_1(a)), \\ \Gamma_2 :& \quad \vec{X}_2(a) = \vec{X}(a, T_2(a)). \end{aligned} \tag{6}$$

What are these curves? Well, they are simply some curves on the tube, further down the tube from the original curve, and Γ_2 is further down than Γ_1. They are otherwise completely arbitrary. The idea of using two different "time" functions is simply to point out that one can shift each point from the original curve $\vec{X}_0(a)$ an arbitrary amount along the integral curves, as long as the new curves remain smooth.

Let's use Λ_1 to denote the surface which is the side of the tube from Γ_0 to Γ_1, and Λ_2 to denote the surface which is the side of the tube from Γ_0 to Γ_2. Clearly

$$\Lambda_1 \subset \Lambda_2. \tag{7}$$

Finally, let Σ_1 be some open surface with boundary Γ_1 and Σ_2 be some other open surface with boundary Γ_2.

Together $\{\Lambda_1, \Sigma_1\}$ form the boundaries of an open tube which extends from Γ_0, along the integral curves of $\vec{v}(\vec{x})$ to the curve Γ_1; let's call this surface $B_1 = \Lambda_1 \cup \Sigma_1$. The tube is open at Γ_0 and closed at Γ_1. Similarly $\{\Lambda_2, \Sigma_2\}$ form the boundaries of an open tube which extends from Γ_0, along the integral curves of $\vec{v}(\vec{x})$ to the curve Γ_2; let's call this surface $B_2 = \Lambda_2 \cup \Sigma_2$. The tube is open at Γ_0 and closed at Γ_2.

3 The vector potential

Let us consider the integral

$$\begin{aligned}\int\int_{B_1} \vec{v}\cdot\hat{n}\, d\sigma &= \int\int_{\Sigma_1} \vec{v}\cdot\hat{n}\, d\sigma + \int\int_{\Lambda_1} \vec{v}\cdot\hat{n}\, d\sigma \\ &= \int\int_{\Sigma_1} \vec{v}\cdot\hat{n}\, d\sigma,\end{aligned} \tag{8}$$

where the second equality holds because $\vec{v}$ is tangent to the surface Λ_1. The same is true for B_2

$$\int\int_{B_2} \vec{v}\cdot\hat{n}\, d\sigma = \int\int_{\Sigma_2} \vec{v}\cdot\hat{n}\, d\sigma. \tag{9}$$

We also see that the portion of this tube from Σ_1 to Σ_2 is a closed volume, lets call it D. Notice that the boundary of D, ∂D, consists of the surface Σ_2, the surface Σ_1 with its normal vector oppositely oriented, and the side surface $\Lambda_2 - \Lambda_1$. Using the divergence theorem, we can express the difference of the two fluxes as

$$\begin{aligned}\int\int_{\partial D} \vec{v}\cdot\hat{n}\, d\sigma &= \int\int\int_D \nabla\cdot\vec{v}\, dV \\ \int\int_{\Sigma_2} \vec{v}\cdot\hat{n}\, d\sigma + \int\int_{\Lambda_2-\Lambda_1} \vec{v}\cdot\hat{n}\, d\sigma + \int\int_{\Sigma_1} \vec{v}\cdot[-\hat{n}]\, d\sigma &= \int\int\int_D \nabla\cdot\vec{v}\, dV \\ \int\int_{\Sigma_2} \vec{v}\cdot\hat{n}\, d\sigma + \int\int_{\Lambda_2} \vec{v}\cdot\hat{n}\, d\sigma - \int\int_{\Lambda_1} \vec{v}\cdot\hat{n}\, d\sigma - \int\int_{\Sigma_1} \vec{v}\cdot\hat{n}\, d\sigma &= \int\int\int_D \nabla\cdot\vec{v}\, dV.\end{aligned} \tag{10}$$

So we conclude that for any vector field and surfaces constructed in this manner,

$$\int\int_{B_2} \vec{v}\cdot\hat{n}\, d\sigma = \int\int_{B_1} \vec{v}\cdot\hat{n}\, d\sigma + \int\int\int_D \nabla\cdot\vec{v}\, dV \tag{11}$$

This means that the two flux integrals in (11) will be equal for $B_{1,2}$ if and only if the divergence of $\vec{v}(\vec{x})$ vanishes everywhere in D. This is the crucial result, i.e. if the divergence is everywhere zero, then the two fluxes are equal. Conversely, if the two fluxes are equal for any B_1 and B_2 constructed in the way we did, then the divergence must vanish on a simply connected set, D.

Now we explain how this applies to the vector potential. Introduce a vector function, $\vec{\Psi}$, which will be the vector potential in a moment. We can consider its line integral around the curve Γ_0, which by Stokes' theorem is

$$\int_{\Gamma_0} \vec{\Psi}\cdot\hat{T}\, dl = \int\int_{B_1} \left(\nabla\times\vec{\Psi}\right)\cdot\hat{n}\, d\sigma. \tag{12}$$

However, B_2 shares the same boundary so that

$$\int_{\Gamma_0} \vec{\Psi}\cdot\hat{T}\, dl = \int\int_{B_2} \left(\nabla\times\vec{\Psi}\right)\cdot\hat{n}\, d\sigma. \tag{13}$$

Therefore

$$\int\int_{B_2} \left(\nabla \times \vec{\Psi}\right) \cdot \hat{n}\; d\sigma = \int\int_{B_1} \left(\nabla \times \vec{\Psi}\right) \cdot \hat{n}\; d\sigma. \tag{14}$$

Using equation (14) and the fact that (11) is an if and only if statement, we find that any vector function $\vec{v}(\vec{x})$ whose divergence is zero must be expressible as a curl.

Now we understand why $\nabla \cdot \vec{v}$ must be zero over a simply connected domain; it is needed in order to be able to integrate the flux in order to compute the vector potential. If we let $\vec{v} = \nabla \times \vec{\Psi}$, we compute the vector potential from the following integral

$$\begin{aligned} \int_{\Gamma_0} \vec{\Psi} \cdot \hat{T}\; dl &= \int\int_{B_1} \vec{v} \cdot \hat{n}\; d\sigma \\ &= \int\int_{\Sigma_i} \vec{v} \cdot \hat{n}\; d\sigma, \end{aligned} \tag{15}$$

for $i = 1$ or 2, or any such surface, for that matter - for example, Σ_0, also.

4 An intuition for this result

I went through this argument because I wanted to re-think the vector potential. One spends so much time thinking about the 2-dimensional case that I think it gives us the wrong intuition. In two dimensions when $\nabla \cdot \vec{v} = 0$ we tend to think of a stream function $\psi(x, y)$ which is made into a vector potential by

$$\vec{v} = \nabla \times \left(-\psi \hat{k}\right) = -\psi_y \hat{\imath} + \psi_x \hat{\jmath}. \tag{16}$$

This is convenient, of course, because contours of constant ψ are stream lines, equivalently integral curves of the vector field.

But the result in equation (15) suggests that we should look at the vector potential differently. The vector potential is a vector field which is best thought of as curving around the velocity field. This is completely intuitive when we recall that the velocity field, itself, can be interpreted as the vector potential of the vorticity field in fluid dynamics. We always think of the velocity as circulating around the integral lines of vorticity.

4.1 Example, vector field in one direction, azimuthal vector potential

The most straightforward example I can think of is a velocity field purely in the z-direction and azimuthally symmetric

$$\vec{v} = f(r)\hat{z} \tag{17}$$

where r is the distance from the z-axis (in cylindrical coordinates) and θ is the angle in that plane. Let Γ be a circle of radius R centered at $r = 0$ so that Σ is the disk of radius R, and $\hat{T} = \hat{\theta}$, is in the azimuthal direction. The normal vector to the plane is $\hat{k}$ and the area element $d\sigma$ is simply the area element in the plane, dA. The vector potential is purely azimuthal and can be written

$$\vec{\Psi} = \lambda(r)\, \hat{\theta} \tag{18}$$

Substituting into equation (15) we write

$$
\begin{aligned}
\int_\Gamma \vec{\Psi}\cdot\hat{T}\,dl &= \int\int_\Sigma \vec{v}\cdot\hat{n}\,d\sigma \\
\int_0^{2\pi} \lambda(R)\hat{\theta}\cdot\hat{\theta}\,Rd\theta &= \int_0^{2\pi}\int_0^R f(r)\,\hat{z}\cdot\hat{z}\,r\,dr\,d\theta \\
R\,\lambda(R) &= \int_0^R f(r)\,r\,dr \\
\Longrightarrow \lambda(r) &= \frac{1}{r}\int_0^r f(r')\,r'\,dr'
\end{aligned}
\tag{19}
$$

This is, of course, what we would do for a vorticity flux tube if we let $\vec{v}$ be the *vorticity* and $\vec{\Psi}$ be the corresponding velocity.

4.2 Example, shear flow in the plane

Let

$$\vec{v} = U(y)\hat{i} = \alpha\,y\,\hat{i}. \tag{20}$$

We can quickly calculate that

$$\vec{\Psi} = \frac{\alpha y^2}{2}\,\hat{k}. \tag{21}$$

Notice that the vector potential is pointing along surfaces of constant $\vec{v}$.

We should also be able to use the integral expression to calculate the vector potential. Let Σ be the rectangular portion of the plane whose normal is $\hat{i}$ between $0 \le y \le a$ and $0 \le z \le b$. Therefore

$$
\begin{aligned}
\int_{\partial\Sigma} \vec{\Psi}\cdot\hat{T}\,dl &= \int_0^a\int_0^b \alpha\,y\,dz\,dy \\
&= \frac{1}{2}\alpha a^2 b
\end{aligned}
\tag{22}
$$

Now at this point we can say much about the direction of $\vec{\Psi}$, but we know that it is unique upto the addition of the gradient of any scalar function. For simplicity we can let $\vec{\Psi} = \lambda(y)\,\hat{z}$ so that

$$
\begin{aligned}
2\alpha a^2 b &= \int_0^b \lambda(a)\,\hat{T}(a)\cdot\hat{z}\,dz + \int_0^b \lambda(0)\,\hat{T}(0)\cdot\hat{z}\,dz \\
&= \int_0^b \lambda(a)\,dz - \int_0^b \lambda(0)\,dz \\
\frac{1}{2}\alpha a^2 b &= b\,[\lambda(a) - \lambda(0)] \\
\lambda(a) &= \frac{1}{2}\alpha a^2 + \lambda(0) \\
\lambda(y) &= \frac{1}{2}\alpha y^2 + \lambda(0)
\end{aligned}
\tag{23}
$$

meaning

$$\vec{\Psi} = \frac{1}{2}\alpha y^2 \hat{k} + \lambda(0)\hat{k}. \tag{24}$$

This is the same as the first expression plus ∇g where $g = \lambda(0)\, z$.

4.3 A simple vortex

Consider the vortex with constant vorticity

$$\vec{v}(\vec{x}) = \frac{\omega}{2}\left[-y\hat{\imath} + x\hat{\jmath} + 0\hat{k}\right] \tag{25}$$

We usually construct the stream function

$$\psi = \frac{\omega}{4}\left[x^2 + y^2\right] \tag{26}$$

so that

$$\vec{v}(\vec{x}) = \nabla \times \left(-\psi\hat{k}\right) = -\psi_y\hat{\imath} + \psi_x\hat{\jmath} + 0\hat{k}. \tag{27}$$

But if we were to extend this velocity field to 3 dimensions, we are really thinking of a velocity field which is invariant in the z direction.

There are other ways to extend this to 3-D, for example, we have a velocity field which is only pointing in the x, y plane, and agrees with the velocity field $\vec{v}$ only at $z = 0$. Let's think about how this would work. What I'm thinking about here is a vector potential that, when expressed in cylindrical coordinates, points in the (r, z) direction and does not depend on θ,

$$\vec{\Psi} = \alpha(r, z)\hat{r} + \beta(r, z)\hat{k}. \tag{28}$$

Taking its curl we must solve

$$\begin{aligned} \vec{v} &= \left[\frac{\partial\alpha}{\partial z} - \frac{\partial\beta}{\partial r}\right]\hat{\theta} \\ \frac{\omega r}{2}\,\hat{\theta} &= \left[\frac{\partial\alpha}{\partial z} - \frac{\partial\beta}{\partial r}\right]\Bigg|_{z=0}\hat{\theta} \end{aligned} \tag{29}$$

in the x, y plane. Of course this does not give a unique solution for α or β. The traditional one used above has

$$\alpha = 0, \quad \beta = -\frac{\omega}{4}\, r^2 \tag{30}$$

It is divergence free, but it is not the only one which is divergence free. We can also set

$$\alpha = \frac{\omega r z}{2}, \quad \beta = 0 \tag{31}$$

and have a purely axially outward pointing vector potential. These both yield the same vector field, though the second one is not divergence free.

4.3.1 A vortex with variable rotation as a function of z

As a thought experiment, Let's extend $\vec{v}$ in a divergence free manner to the whole of $\mathbf{R}^3$

$$\vec{v} = \frac{\omega r}{2}\, f(z)\, \hat{\theta} \quad \text{where} \quad f(0) = 1. \tag{32}$$

Now we want to find a vector potential which has the form (28) and whose curl gives us $\vec{v}$. One such solution is

$$\alpha = 0, \quad \beta = -\frac{\omega}{4}\, f(z)\, r^2. \tag{33}$$

another solution is

$$\alpha = \frac{\omega r}{2}\left[C + \int_0^z f(z')\, dz'\right] \tag{34}$$

for any constant C. Now we can take the average of these two vector potentials and add a gauge, $G(r, z)$ to attain the most general solution

$$\vec{\Psi} = \frac{\omega}{4}\left[\left(r\int_0^z f(z')\, dz'\right)\hat{r} - \frac{r^2 f(z)}{2}\,\hat{k} + \nabla G,\right] \tag{35}$$

where we recognize that the constant term in the $\hat{r}$ component can be incorporated into the gauge. A natural constraint is to make $\vec{\Psi}$ divergence free which implies

$$\nabla^2 G + 2\int_0^z f(z')\, dz' - \frac{r^2}{2}\frac{df}{dz} = 0 \tag{36}$$

It is convenient to define 3 antiderivatives of f as $F(z)$. Therefore

$$\nabla^2 G + 2F'' - \frac{r^2}{2}F'''' = 0 \tag{37}$$

let's assume a solution

$$G = G_0(z) + G_2(z)r^2 \quad \Longrightarrow \nabla^2 G = G_0'' + G_2'' r^2 + 4G_2. \tag{38}$$

Therefore

$$G_2 = \frac{F''}{2} \tag{39}$$

and

$$\begin{aligned} G_0'' + 4G_2 + 2F'' &= 0 \\ G_0'' + 4F'' &= 0 \\ G_0 &= -4F \end{aligned} \tag{40}$$

therefore

$$G = \frac{r^2}{2}F'' - 4F. \tag{41}$$

Writing $\vec{\Psi}$ in terms of $F(z)$ we find

$$\begin{aligned}
\vec{\Psi} &= \frac{\omega}{4}\left[rF''\,\hat{r} - \frac{r^2F'''}{2}\,\hat{k} + rF''\,\hat{r} + \frac{r^2}{2}\,F'''\,\hat{k} - 4F'\,\hat{k}\right] \\
&= \frac{\omega}{4}\left[2rF''\,\hat{r} - 4F'\,\hat{k}\right] \\
&= \frac{\omega}{2}\left[r\hat{r}\int_0^z f(z')\,dz' - 2\hat{k}\int_0^z\int_0^{z'} f(z'')\,dz''\,dz'\right]
\end{aligned} \tag{42}$$

In this way we get a unique, divergence-free vector potential that is clearly not completely in the $\hat{k}$ direction. However, I have also extended the velocity in the plane to some non-constant velocity in the 3-Dimensional space that, while it agrees with the original velocity in the x, y plane, it decays away from the plane. Of course, since $\vec{\Psi}$ is divergence-free, we can create its integral curves $\vec{X}(a,t) = R(a,t)\hat{r} + Z(a,t)\hat{k}$ by solving

$$\begin{aligned}
\frac{dR}{dt} &= \frac{\omega}{2}RF''(R,Z) = \frac{\omega}{2}\frac{\partial}{\partial Z}\left(RF'\right) \\
\frac{dZ}{dt} &= -\frac{\omega}{2}\left(2F'(R,Z)\right) = -\frac{\omega}{2}\frac{1}{R}\frac{\partial}{\partial R}\left[R\left(RF'\right)\right].
\end{aligned} \tag{43}$$

So plotting contours of constant $rF'(z)$ in the (r,z) plane will yield the integral curves of $\vec{\Psi}$ in this formulation. Since I need a clear example to understand these concepts, I will choose

$$\begin{aligned}
& f(z) = e^{-\frac{|z|}{H}} \\
\Longrightarrow & \int_0^z f(z')\,dz' = H\sigma(z)\left[1 - e^{-\frac{|z|}{H}}\right] \quad \text{where } \sigma(z) \text{ is the signum function} \\
\Longrightarrow F'(z) &= \int_0^z\int_0^{z'} f(z'')\,dz''\,dz' \\
&= \int_0^z H\sigma(z')\left[1 - e^{-\frac{|z'|}{H}}\right]\,dz' \\
&= H|z| + H^2\left[e^{-\frac{|z|}{H}} - 1\right] \\
&= H^2\left[\frac{|z|}{H} + e^{-\frac{|z|}{H}} - 1\right].
\end{aligned} \tag{44}$$

In figure 1 I have plotted a slice of the (r,z) plane at constant θ showing contours of constant $rF'(z)$, which are the integral curves of $\vec{\Psi}$.

4.3.2 Returning to the constant rotation vortex

Consider also the limit that $H \to \infty$ of the expression in (44). In this case

$$rF'(z) \to \frac{rz^2}{2} \tag{45}$$

and

$$\vec{\Psi}_* = \frac{\omega}{4}\left[2rz\hat{r} - 2z^2\hat{k}\right] = \frac{\omega}{2}\,z\,\left[r\hat{r} - z\hat{k}\right]. \tag{46}$$

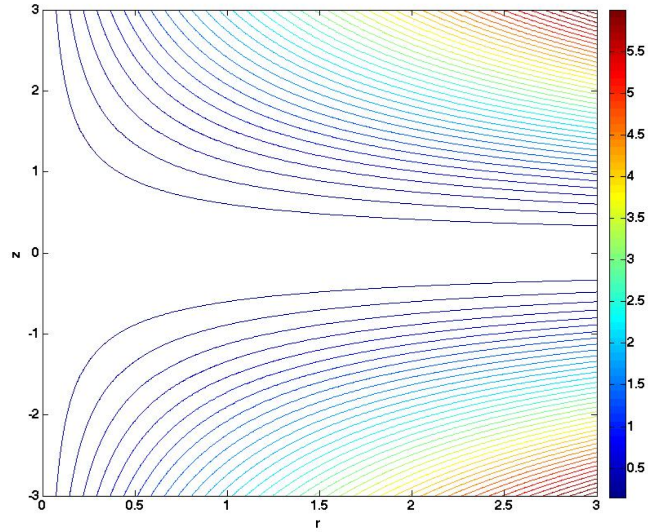

Figure 1: The (r, z) plane at constant θ showing integrals curves of $\vec{\Psi}$. The vectors are pointing away from the axis in the upper half plane and toward the axis in the lower half plane.

This is a vector potential for a z-invariant, infinite in the (x, y)-plane vortex since the curl of this vector potential is

$$\nabla \times \vec{\Psi}_* = \frac{\omega}{2} r \hat{\theta}, \tag{47}$$

which is the vortex velocity field. Notice that this differs from the traditional vector potential by

$$\begin{aligned} \vec{\Psi}_* - \left(-\psi \hat{k}\right) &= \frac{\omega}{2}\ z\ \left[r\hat{r} - z\hat{k}\right] + \frac{\omega}{4} r^2 \hat{k} \\ &= \frac{\omega}{4} \left[2rz\hat{r} + \left(r^2 - 2z^2\right) \hat{k}\right] \\ &= \frac{\omega}{4}\ \nabla \left(r^2 z - \frac{2z^3}{3}\right), \end{aligned} \tag{48}$$

which is a gradient of a scalar function. In fact, since the divergence of both terms on the left hand side of (48) are equal to zero, it is clear that the Laplacian of the scalar function must also be equal to zero. Let's check

$$\begin{aligned} \nabla^2 \left[r^2 z - \frac{2z^3}{3}\right] &= \nabla^2 \left[x^2 z + y^2 z - \frac{2z^3}{3}\right] \\ &= 2z + 2z - 4z \\ &= 0 \end{aligned} \tag{49}$$

This makes me ask the question, what is the most general cubic function with zero

Laplacian and cylindrical symmetry? Let's write

$$\begin{aligned}
G(r,z) &= Ar^3 + Br^2z + Crz^2 + Dz^3 \\
\nabla^2 G &= \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\frac{\partial}{\partial r}\left[Ar^3 + Br^2z + Crz^2 + Dz^3\right]\right\} + \frac{\partial^2}{\partial z^2}\left[Ar^3 + Br^2z + Crz^2 + Dz^3\right] \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left[3Ar^3 + 2Br^2z + Crz^2\right] + 2Cr + 6Dz \\
&= 9Ar + 4Bz + C\frac{z^2}{r} + 2Cr + 6Dz.
\end{aligned} \tag{50}$$

For this to be zero for all (r,z) we find $C = A = 0$, and $4B + 6D = 0$ which means if $B = 1$, $D = -\frac{2}{3}$, and

$$G = r^2z - \frac{2z^3}{3} \tag{51}$$

which is the solution we already found.

5 Interpretation

My point here is that the intuition one gleans from 2D - i.e. from using the Cauchy Riemann equations or as Hafez suggests, using Gauss' theorem to calculate the vector potential is not quite correct. In fact, it is Stoke's theorem that is used to calculate the vector potential (as I showed above). The correct intuition for the vector potential is that it tends to curve around the integral curves of the vector field for which it is the potential.

This is not the way we really think of things when we use a stream function formulation in 2D to act as the "vector potential" of an incompressible flow. In that instance, the vector potential is purely in the $\hat{k}$ direction, it does not curve around the integral curves of the velocity vector field.

Similarly I used a simple (unbounded) vortex in 2D to demonstrate this result. By thinking of the vortex as extended infinitely in the z-direction, we get a vector potential which either points completely in the z-direction, or one that points away from the axis.