1 Derivation of Wave Equation

Consider an elastic under tension of length L. Fix the endpoints at $\vec{x} = 0$ and $\vec{x} = L\hat{i}$. The mass density of the of the elastic is ρ, a constant and its cross sectional area is A. If we consider segments of equal mass Δm_i then Newton's second law for the i^{th} mass is

$$\Delta m_i \frac{d^2\vec{x}_i}{dt^2} = \vec{F}(x_i) \tag{1}$$

The length of each segment is ΔL_i and we can write $\Delta m_i = \rho A \Delta L_i$ where ΔL_i is the length of the string in its rest state. We will consider segments of equal mass so that

$$\Delta m_i = \Delta m = \rho A \Delta L = \frac{\rho A L}{n} \tag{2}$$

where n is the number of segments (which we will eventually make go to infinity). The i^{th} piece of the string is located at $\vec{x}_i$.

Now, let us consider the force on this piece of string. It is only affected by its nearest neighbors and is in the direction of the vector point from i to the nearest neighbor. We will assume the strength of the force is only a function of the distance between the two points, i.e. only a function of the length of that segment. Therefore the force on the i^{th} piece from the $i+1^{th}$ piece is

$$\vec{F}_{i+1,i} = T\left(\frac{|\vec{x}_{i+1} - \vec{x}_i|}{\Delta L}\right) \frac{\vec{x}_{i+1} - \vec{x}_i}{|\vec{x}_{i+1} - \vec{x}_i|} \tag{3}$$

similarly for the mass to the right of the i^{th} mass

$$\vec{F}_{i-1,i} = T\left(\frac{|\vec{x}_{i-1} - \vec{x}_i|}{\Delta L}\right) \frac{\vec{x}_{i-1} - \vec{x}_i}{|\vec{x}_{i-1} - \vec{x}_i|} \tag{4}$$

where we have made the magnitude depend on the ratio of the length of the segment at any given time compared to its length when it is at rest - this is the stretch factor. Let's call this factor

$$r_i = \frac{|\vec{x}_{i-1} - \vec{x}_i|}{\Delta L} \tag{5}$$

The magnitude of the force is called the **tension** and in order for there to be any forces on the elastic, there must be tension even when the elastic is not stretched (i.e. laid out flat between its endpoints). Let r be this ratio, therefore

$$T(r) = T_0\, r\, f(r) \tag{6}$$

where $f(1) = 1$, T_0 is the tension in the elastic when it is not stretched and $f(r)$ is a function of the stretch factor. Usually $f(r)$ is monotonically increasing, but it could generally be a complicated function of r, depending on the medium. If there is an $r = r_{\text{snap}}$ for which the elastic snaps, then $f(r) = 0$ for $r \geq r_{\text{snap}}$. You may ask, "why did we choose such a particular form for the tension"? The answer is Taylor series. The conditions we put on the tension, i.e. that it must be a function of the stretch factor and it must be equal to T_0 when the string is at rest, make this functional form the only option. If $f(r)$ is equal to 1 for all r, then this is called a linear elastic.

So let's write Newton's law for the i^{th} segment, define

$$\begin{aligned} \rho A \Delta L \frac{d^2 \vec{x}_i}{dt^2} &= \frac{T(r_{i+1})}{r_{i+1}\Delta L}\left[\vec{x}_{i+1} - \vec{x}_i\right] + \frac{T(r_i)}{r_i \Delta L}\left[\vec{x}_{i-1} - \vec{x}_i\right] \\ \rho A \frac{d^2 \vec{x}_i}{dt^2} &= \frac{T_0}{\Delta L^2}\left\{ f(r_{i+1})\left[\vec{x}_{i+1} - \vec{x}_i\right] - f(r_i)\left[\vec{x}_i - \vec{x}_{i-1}\right]\right\}. \end{aligned} \tag{7}$$

We are using a parameterization of the elastic in which each piece has equal mass. This motivates the notion of a *Lagrangian* coordinate, which is a coordinate that follows the piece of string

$$a_i = i\ \Delta L \quad \text{for} \quad i = 1 \ldots n. \tag{8}$$

We use this to define a continuum version of the position vector $\vec{x}(a,t)$ where

$$\vec{x}(a_i, t) = \vec{x}_i(t). \tag{9}$$

Therefore in the limit $\Delta L \to 0$

$$\frac{\vec{x}_i - \vec{x}_{i-1}}{\Delta L} \to \frac{\partial \vec{x}}{\partial a}(a,t), \quad r_i \to \left|\frac{\partial \vec{x}}{\partial a}(a,t)\right|. \tag{10}$$

Making these substitutions into the right hand side of the momentum equation and recognizing that the time derivative must be changed to the partial time derivative we find the nonlinear wave equation

$$\rho A \frac{\partial^2 \vec{x}}{\partial t^2} = T_0 \frac{\partial}{\partial a}\left[f\left(\left|\frac{\partial \vec{x}}{\partial a}\right|\right)\frac{\partial \vec{x}}{\partial a}\right] \tag{11}$$

Notice that a has units of length (as does $\vec{x}$). The ratio

$$\frac{T_0}{\rho A} = c^2 \tag{12}$$

has units of speed squared.

2 Conservation of energy

Take the dot product of equation (11) with $\frac{\partial \vec{x}}{\partial t}$ and integrate over the domain $0 \le a \le L$ to yield

$$\begin{aligned} \int_0^L \rho A\ \vec{x}_t \cdot \vec{x}_{tt}\ da &= \int_0^L T_0\ \vec{x}_t \cdot \frac{\partial}{\partial a}\left[f\left(\left|\frac{\partial \vec{x}}{\partial a}\right|\right)\frac{\partial \vec{x}}{\partial a}\right]\ da \\ \frac{d}{dt}\int_0^L \frac{\rho A}{2}\ |\vec{x}_t|^2\ da &= T_0\left[\vec{x}_t \cdot \vec{x}_a\ f\left(|\vec{x}_a|\right)|_{x=0}^{x=L} - \int_0^L f(|\vec{x}_a|)\ \vec{x}_{a\,t} \cdot \vec{x}_a\ da\right] \end{aligned} \tag{13}$$

etc...

3 Linear Initial Value problem

When $f(r) = 1 \forall r$ the wave equation becomes

$$\vec{x}_{tt} = c^2 \vec{x}_{aa} \tag{14}$$

and each of the three dimensions of motion is not coupled to the other, i.e. let $\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$x_{tt} = c^2 x_{aa}, \quad y_{tt} = c^2 y_{aa}, \quad z_{tt} = c^2 z_{aa} \tag{15}$$