The effect of linear drag on a particle in a constant gravitational force.

May 4, 2014

1 Derivation

Consider a particle in a gravitational field where the acceleration of gravity acts in the downward, $-\hat{k}$ direction. The force of gravity is

$$\vec{F}_{grav} = -mg\hat{k}.$$

The drag force always acts to oppose the velocity and we will assume a linear drag law, i.e.

$$\vec{F}_{drag} = -C\vec{v}$$

where μ is called the drag coefficient. Newton's second law states that

$$\vec{a} = \frac{\vec{F}}{m}$$

where the force on the right hand side is the sum of all the forces acting on the object and the acceleration vector is the time derivative of the velocity vector. If we substitute the sum of gravitational and drag forces we find

$$\frac{d\vec{v}}{dt} = -g\hat{k} - \frac{\vec{v}}{\tau} \tag{1}$$

where

$$\tau = \frac{m}{C}$$

is the drag rate and it has units Time^{-1}.

In component form, equation (1) becomes

$$\frac{du}{dt}\hat{\imath} + \frac{dv}{dt}\hat{\jmath} + \frac{dw}{dt}\hat{k} = -g\hat{k} - C\left(u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right).$$

where, now I am using u, v, w to denote the x, y, z components of the velocity. In order to solve this system of equations we take the dot product of the equation with each of the unit vectors; the x-component

$$\frac{du}{dt} = -\frac{u}{\tau} \tag{2}$$

the y-component

$$\frac{dv}{dt} = -\frac{v}{\tau} \tag{3}$$

and the z-component

$$\frac{dw}{dt} = -g - \frac{w}{\tau}. \tag{4}$$

Let us solve the first equation by dividing by u and integrating with respect to time

$$\int_0^t \frac{1}{u}\frac{du}{dt'}dt' = -\frac{1}{\tau}\int_0^t dt' \tag{5}$$

Now we use chain rule (which is u-substitution) on the integral on the left hand side.

$$du = \frac{du}{dt'}dt', \quad \text{when} \quad t' = 0, u = u(0); \;\; t' = t, u = u(t) \tag{6}$$

so that the equation simplifies to

$$\int_{u_0}^{u(t)} \frac{1}{u}du = -\frac{t}{\tau} \tag{7}$$

where I have also simplified the right hand side. Evaluating the left hand side we find

$$\ln\left|\frac{u(t)}{u_0}\right| = -\tfrac{t}{\tau} \tag{8}$$

$$\therefore u(t) = u_0\, e^{-\frac{t}{\tau}}. \tag{9}$$

By the same argument one can show that

$$v(t) = v_0\, e^{-\frac{t}{\tau}}. \tag{10}$$

We can perform the same series of steps to solve equation (4), except the first integral becomes

$$\int_0^t \frac{1}{g\tau + w}\frac{dw}{dt'}dt' = -\frac{1}{\tau}\int_0^t dt' \tag{11}$$

(Notice that I factored out a C from the right hand side of (4) before I divided both sides by $g\tau + w$.) Now we use chain rule (which is u-substitution) on the integral on the left hand side.

$$dw = \frac{dw}{dt'}dt', \quad \text{when} \quad t' = 0, w = w_0;\;\; t' = t, w = w(t) \tag{12}$$

so that the equation simplifies to

$$\int_{w_0}^{w(t)} \frac{1}{g\tau + w}dw = -\frac{t}{\tau} \tag{13}$$

where I have again simplified the right hand side. Again, the left hand side evaulates to a natural logarithm

$$\begin{aligned} \ln\left|\frac{w(t) + g\tau}{w(0) + g\tau}\right| &= -\frac{t}{\tau} \\ \frac{w(t) + g\tau}{w_0 + g\tau} &= e^{-\frac{t}{\tau}} \\ \therefore w(t) &= (w_0 + g\tau)\; e^{-\frac{t}{\tau}} - g\tau. \end{aligned}$$

Notice that after a long time

$$\lim_{t\to\infty} w(t) = -g\tau$$

which is to say, the downward speed approaches a constant at long times. This constant, $g\tau$ is called the terminal velocity.

Now we'd like to calculate $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ by integrating

$$\frac{d\vec{r}}{dt} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \tag{14}$$

at this point we have already calculated the three components of the velocity. Substituting the expression for the velocities into equation (14) it is a simple calculation to show (you should show this yourself)

$$\begin{aligned} \frac{dx}{dt} &= u_0 e^{-\frac{t}{\tau}} \quad \Longrightarrow \quad x(t) = x_0 + u_0 \tau \left[1 - e^{-\frac{t}{\tau}}\right] \\ \frac{dy}{dt} &= v_0 e^{-\frac{t}{\tau}} \quad \Longrightarrow \quad y(t) = y_0 + v_0 \tau \left[1 - e^{-\frac{t}{\tau}}\right] \end{aligned} \tag{15}$$

while for the vertical position we find

$$\begin{aligned} \frac{dz}{dt} &= (w_0 + g\tau)\ e^{-\frac{t}{\tau}} - g\tau \\ \Longrightarrow z(t) &= z_0 + \tau \left[(w_0 + g\tau)\ \left(1 - e^{-\frac{t}{\tau}}\right) - gt\right] \end{aligned} \tag{16}$$

O.k., these are nice algebraic expressions, but we should really think about them some more to see what the curves look like. Let us consider the case of a projectile moving in the (x, z) plane that begins at the origin. Therefore we set $x_0 = y_0 = z_0 = v_0$ so that $y(t) = 0$ for all time. Let us let the initial speed be $u_0 \hat{i} + w_0 \hat{k} = s \left(\cos(\alpha)\hat{i} + \sin(\alpha)\hat{k}\right)$, where s is the initial speed and α is the initial angle with respect to the horizontal. The expressions for x and z become

$$\begin{aligned} x(t) &= s\tau\ \cos(\alpha)\ \left[1 - e^{-\frac{t}{\tau}}\right] \\ z(t) &= \left(s\tau\ \sin(\alpha) + g\tau^2\right)\ \left(1 - e^{-\frac{t}{\tau}}\right) - g\tau t \end{aligned} \tag{17}$$

An interesting exercise is to assume the drag is small, i.e. that τ is very very large - this should reproduce the solution with no drag. In this case, if you substitute

$$e^{-\frac{t}{\tau}} = 1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} + ... \tag{18}$$

into equation (17) and take the limit $\tau \to \infty$ you will find

$$\begin{aligned} x(t) &= st\cos(\alpha) \\ z(t) &= st\ \sin(\alpha) - \frac{gt^2}{2} \end{aligned} \tag{19}$$

which is the solution (a parabola) that we found in class for the case without drag.

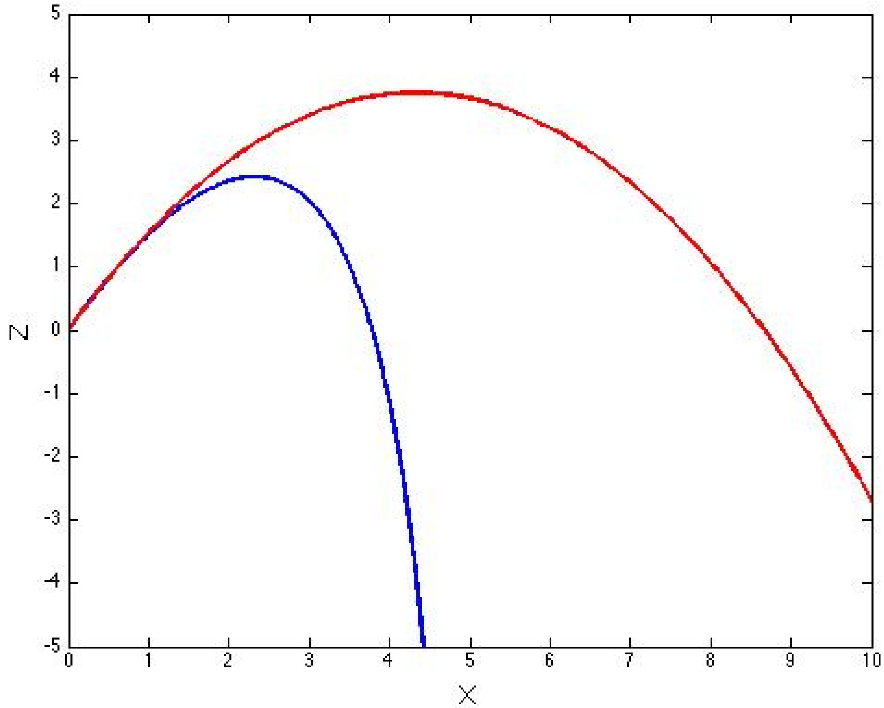

Figure 1: Comparison of two particles fired upward. Blue curve has the particle under drag, while the red curve has no drag. The red curve is a parabola, while the blue curve is close to the red curve at the beginning, but deviates sharply afterwards.

The attached figure shows two particles fired with initial speed of 10 m/s at an angle of 60° from the horizontal (i.e. $\pi/3$). The gravitational acceleration is $g = 10\,\mathrm{m/s^2}$. The red curve has no drag, i.e. $\tau \to \infty$. The blue curve uses a drag time of 1 second.

2 Nondimensional version

A nondimensionalization of this problem would work as follows. Let

$$t = \frac{s}{g}\tilde{t}, \quad (x, z) = \frac{s^2}{g}(\tilde{x}, \tilde{z}) \tag{20}$$

and the nondimensional drag rate is

$$\epsilon = \frac{s}{\tau g} \tag{21}$$

which is the ratio of the initial speed to the terminal velocity. The nondimensional solution is

$$\begin{aligned} x &= \cos(\alpha)\left[\frac{1 - e^{-\epsilon t}}{\epsilon}\right] \\ z &= \left(\sin(\alpha) + \frac{1}{\epsilon}\right)\left[\frac{1 - e^{-\epsilon t}}{\epsilon}\right] - \frac{t}{\epsilon} \end{aligned} \tag{22}$$

where I have dropped the tildes.

The small ϵt limit of these expressions is

$$\begin{aligned} x &= t\cos(\alpha)\left[1 - \frac{\epsilon t}{2} + ...\right] \\ z &= \left(\sin(\alpha) + \frac{1}{\epsilon}\right)\left[t - \frac{\epsilon t^2}{2} + ...\right] - \frac{t}{\epsilon} \\ &= t\sin(\alpha)\left[1 - \frac{\epsilon t}{2} + ...\right] - \frac{t^2}{2}\left[1 - \frac{\epsilon t}{3} + ...\right]. \end{aligned} \tag{23}$$

and the $\epsilon = 0$ limit is manifestly a parabola.

Now we can solve for t in terms of x

$$t = -\frac{1}{\epsilon}\ln\left|1 - \frac{\epsilon x}{\cos(\alpha)}\right| \tag{24}$$

so that the explicit expression for $z(x)$ is

$$z = \left(\sin(\alpha) + \frac{1}{\epsilon}\right)\frac{x}{\cos(\alpha)} + \frac{1}{\epsilon^2}\ln\left|1 - \frac{\epsilon x}{\cos(\alpha)}\right| \tag{25}$$

Notice that as $t \to \infty$ the maximum value of x is

$$\begin{aligned} x &\longrightarrow \frac{\cos(\alpha)}{\epsilon} \equiv x_{\mathrm{M}} \\ z &\longrightarrow \left(\frac{\sin(\alpha)}{\epsilon} + \frac{1}{\epsilon^2}\right) + \frac{1}{\epsilon^2} \ln\left(1 - \frac{x}{x_{\mathrm{M}}}\right) \end{aligned} \tag{26}$$

One final interesting question is "What is the horizontal distance travelled by a projectile under the influence of gravity and drag?" What I mean by this is, assume the projectile is shot from the ground, rises, and returns to the ground. How big is x when it hits the ground again? So we need to solve $z(t_*) = 0$ for t_* and we find

$$t_* = 2\sin(\alpha) \tag{27}$$

in the case when there is no drag and therefore

$$x(t_*) \equiv x_* = \sin(2\alpha). \tag{28}$$

This clearly shows that the maximum range (for a fixed speed and gravity) occurs when $\alpha = \pi/4$.

In the presence of drag, the expression for x_* is a transcendental equation

$$(\epsilon \sin(\alpha) + 1)\left[\frac{x_*}{x_{\mathrm{M}}}\right] = -\ln\left[1 - \left(\frac{x_*}{x_{\mathrm{M}}}\right)\right]. \tag{29}$$

Defining

$$\delta = \epsilon \sin(\alpha) \quad \text{and} \quad r(\delta) = \frac{x_*}{x_{\mathrm{M}}} \tag{30}$$

then we must solve

$$(1+\delta)\, r = -\ln(1-r)\,. \tag{31}$$

Notice that $r = 0$ is always a solution of equation (31). There is a non-trivial solution near $r = 0$ given, approximately by

$$\begin{aligned} (1+\delta)\, r &\approx r + \frac{r^2}{2} + \frac{r^3}{3} \\ 0 &\approx 2r^2 + 3r - 6\delta \\ \Longrightarrow\ r &\approx 2\delta\left[1 - \frac{4}{3}\delta + \ldots\right] \\ \Longrightarrow\ x_* &\approx \sin(2\alpha)\ \left[1 - \frac{4\epsilon\sin(\alpha)}{3}\right] \\ \Longrightarrow\ x_* &\approx \sin(2\alpha)\ \left(\frac{s^2}{g}\right)\ \left[1 - \frac{4s\sin(\alpha)}{3\tau g}\right] \end{aligned} \tag{32}$$

where I have re-dimensionalized x_* in the last expression. This is the solution for a projectile in the absence of drag plus the first correction due to drag. It is valid in the limit of small ϵ or small α.

For large δ the solution to equation (31) is close to $r = 1$ therefore we let

$$r = 1 - s \tag{33}$$

and find

$$(1 + \delta)(1 - s) = -\ln(s) \tag{34}$$

so that

$$r \approx 1 - e^{-(1+\delta)}. \tag{35}$$

Therefore the range of the projectile is

$$\begin{aligned} x_* &\approx \frac{\cos(\alpha)}{\epsilon}\left[1 - \exp\left[-\left(1 + \epsilon \sin(\alpha)\right)\right]\right] \\ x_* &\approx s\tau\, \cos(\alpha)\left[1 - \exp\left[-\left(1 + \frac{s \sin(\alpha)}{\tau g}\right)\right]\right] \end{aligned} \tag{36}$$

again, the last expression has been re-dimensionalized. This result is valid for large ϵ for all α.