The WKB approximation and Ray Theory

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1 Derivation

We're going to study wave equation in a medium with varying wave speed

$$h_{tt} - \nabla \cdot \left[c^2(\vec{x}) \nabla h \right] = 0 \tag{1}$$

using the WKB approximation. This is a constant coefficients (in time) PDE so we can consider each frequency in isolation

$$h(\vec{x}, t) = e^{-i\omega t} \, F(\vec{x}) \tag{2}$$

leaving the PDE

$$\omega^2 F + \nabla \cdot \left[c^2(\vec{x}) \nabla F \right] = 0. \tag{3}$$

This is sufficiently general to be impossible to deal with using exact methods. We will instead use the approximation that the wave speed changes over length scales which are long compared to the wave number

$$k(\vec{x}) = \frac{\omega}{c(\vec{x})}. \tag{4}$$

Introducing the phase function

$$F = e^{i\Phi(\vec{x})} \tag{5}$$

the PDE becomes

$$\begin{aligned} 1 + e^{-i\Phi} \nabla \cdot \left[\frac{i}{k^2} \nabla \Phi \, e^{i\Phi} \right] &= 0 \\ 1 - \frac{|\nabla \Phi|^2}{k^2} + \nabla \cdot \left[\frac{i}{k^2} \nabla \Phi \right] &= 0. \end{aligned} \tag{6}$$

As I mentioned in class, if $k =$ constant then the solution of this equation is

$$\Phi = k \left(\hat{t} \cdot \vec{x} \right) + \text{constant} \tag{7}$$

where $\hat{t}$ is any unit tangent vector to the ray. This is balance of the first two terms on the right hand side.

The way asymptotics is done is that a small parameter is chosen to describe a ratio of scales in the problem. In this case, there is a length scale associated with the variation of k

$$L_{\text{change}} = \left[\frac{d\ln(k)}{dx}\right]^{-1} \tag{8}$$

and there is a wavelength associated with k

$$L_{\text{wave}} = k^{-1}. \tag{9}$$

In the WKB approximation we assume that we are dealing with waves where

$$\frac{L_{\text{wave}}}{L_{\text{change}}} = \epsilon \ll 1. \tag{10}$$

which also means

$$\begin{aligned} \frac{1}{k} &\ll \left|\frac{1}{k}\frac{dk}{dx}\right|^{-1} \\ \left|\frac{1}{k^2}\frac{dk}{dx}\right| &\ll 1 \\ \left|\frac{d(k^{-1})}{dx}\right| &\ll 1 \\ \left|\frac{d(\lambda)}{dx}\right| &\ll 1 \end{aligned} \tag{11}$$

where λ is the wavelength of the wave. So, another way to think about the WKB approximation is that it works when the wavelength of the wave changes GRADUALLY as the wave propagates through the medium.

Effectively, how this works is that we assume k is very large and Φ is very large, consistent with (7). So, make the replacements

$$\Phi \to \frac{\Phi}{\epsilon}, \quad \text{and} \quad k \to \frac{k}{\epsilon} \tag{12}$$

which makes the PDE become

$$1 - \frac{|\nabla\Phi|^2}{k^2} + \epsilon\, \nabla \cdot \left[\frac{i}{k^2}\nabla\Phi\right] = 0. \tag{13}$$

In asymptotic methods, we seek a solution which is a power series in ϵ. There is a lot to be said about this series, but suffice for now that it is not always a convergent series. In WKB it is convergent. Let

$$\Phi = \Phi_0 + \epsilon\Phi_1 + \epsilon^2 .... \tag{14}$$

If we substitute this asymptotic series into equation (13) and only consider teams unto first order in ϵ we find

$$1 - \frac{|\nabla\Phi_0|^2}{k^2} - \epsilon\left\{\frac{2\nabla\Phi_1 \cdot \nabla\Phi_0}{k^2} - \nabla \cdot \left[\frac{i}{k^2}\nabla\Phi_0\right]\right\} + ... = 0 \tag{15}$$

Now the philosophy is that we want equation (15) to be valid in the limit $\epsilon \to 0$. The only way this can happen is if the coefficient of every power of ϵ is itself zero, therefore we find

$$|\nabla\Phi_0|^2 = k(\vec{x})^2 \quad \text{The Eikonal Equation} \tag{16}$$

and

$$\frac{2\nabla\Phi_1 \cdot \nabla\Phi_0}{k^2} = \nabla \cdot \left[\frac{i}{k^2}\nabla\Phi_0\right] \quad \text{The Amplitude Equation.} \tag{17}$$

We will use the Eikonal equation to construct the ODEs for the rays and we will use the Amplitude equation to describe how the amplitude of the waves changes as the rays propagate.

2 Ray Theory

We will define a ray as the curve which is everywhere perpendicular to the surface of constant phase, Φ. Let τ be an arbitrary parameter for the ray curve and $\vec{r}(\tau) = X_1(\tau)\hat{e}_1 + X_2(\tau)\hat{e}_2 + X_3(\tau)\hat{e}_3$ be vector pointing from the origin to the ray for each value of τ.

Since the ray is perpendicular to the surface of constant Φ then it its tangent vector is parallel to the gradient of Φ

$$\frac{d\vec{r}}{d\tau} = \lambda(\vec{x})\,\nabla\Phi_0 \tag{18}$$

for any arbitrary function $\lambda(\vec{x})$. We need to derive a closed equation for $\vec{r}$, so start by dividing (18) by λ and taking the derivative again

$$\begin{aligned} \frac{d}{d\tau}\left[\frac{1}{\lambda(\vec{x})}\frac{dX_i}{d\tau}\right] &= \frac{d}{d\tau}\frac{\partial\Phi_0}{\partial x_i} \\ &= \frac{dX_j}{d\tau}\frac{\partial}{\partial x_j}\frac{\partial\Phi_0}{\partial x_i} \end{aligned} \tag{19}$$

where we use the convention that repeated indices (i.e. j) are summed over. Now using equation (18) on the right hand side of (19) we find

$$\begin{aligned} \frac{d}{d\tau}\left[\frac{1}{\lambda(\vec{x})}\frac{dX_i}{d\tau}\right] &= \lambda\frac{\partial\Phi_0}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial\Phi_0}{\partial x_i} \\ &= \lambda\frac{\partial\Phi_0}{\partial x_j}\frac{\partial}{\partial x_i}\frac{\partial\Phi_0}{\partial x_j} \\ &= \lambda\frac{\partial}{\partial x_i}\frac{|\nabla\Phi_0|^2}{2} \\ \frac{1}{\lambda}\frac{d}{d\tau}\left[\frac{1}{\lambda}\frac{d\vec{r}}{d\tau}\right] &= \frac{1}{2}\nabla\left(k^2\right) \end{aligned} \tag{20}$$

where I have used equation (16) to simplify the last line. This is the ray equation and it is valid for any choice of λ. It is a second order equation where the right hand side is known, λ can be chosen arbitrarily and $\vec{r}$ is determined from a initial position and initial derivative. So it is exactly like a particle in a potential in Newtonian mechanics. I will solve it in a couple of simple cases after we discuss a conserved quantity of the ray dynamics.

2.1 Energy-like equation for rays

By taking the dot product of (20) with $\frac{d\vec{r}}{dt}$ we can create a conserved quantity for the rays

$$\frac{d}{d\tau}\left[\frac{1}{\lambda^2}\left|\frac{d\vec{r}}{d\tau}\right|^2 - k^2\right] = 0 \tag{21}$$

or

$$\frac{1}{\lambda^2}\left|\frac{d\vec{r}}{d\tau}\right|^2 - k^2 = \alpha \tag{22}$$

where α is a constant of integration. This is exactly what you would do to determine conservation of energy in Newtonian mechanics - the first term being the kinetic energy, the second term being the potential energy and the last term is the constant total energy.

2.2 A choice of λ is a choice of parameterization, τ.

2.2.1 Optical depth parameterization, $\lambda = 1$

Since λ is arbitrary, just make it equal to 1 and the ray equations become

$$\frac{d^2\vec{r}}{d\tau^2} = \nabla\left(\frac{k^2}{2}\right) \tag{23}$$

with the conserved quantity

$$\left|\frac{d\vec{r}}{d\tau}\right|^2 = k^2. \tag{24}$$

The ray equation looks exactly like an equation for a particle in Newtonian mechanics and τ is called the optical depth.

2.2.2 The arc length parameterization, $\lambda = k^{-1}$

One could make

$$\lambda = \frac{1}{|\nabla\Phi_0|} = \frac{1}{k} \tag{25}$$

from the Eikonal equation. This makes the perpendicularity equation

$$\frac{d\vec{r}}{d\tau} = \frac{\nabla\Phi_0}{k}. \tag{26}$$

so that $\frac{d\vec{r}}{d\tau}$ is a unit vector. This means that τ is the arc length along the ray and the ray equations become

$$k\frac{d}{d\tau}\left[k\frac{d\vec{r}}{d\tau}\right] = \frac{1}{2}\nabla\left(k^2\right) \tag{27}$$

with the conserved quantity

$$\left|\frac{d\vec{r}}{d\tau}\right| = 1. \tag{28}$$

2.2.3 The phase parameterization, $\lambda = k^{-2}$

However, what we really want to know is where are the surfaces of constant phase, Φ_0. To figure this out, lets take the derivative

$$\begin{aligned}\frac{d}{d\tau}\Phi_0 &= \frac{d\vec{r}}{d\tau}\cdot\nabla\Phi_0 \\ &= \lambda|\nabla\Phi_0|^2 \qquad \text{by equation (18)} \\ \frac{d}{d\tau}\Phi_0 &= \lambda k^2 \qquad \text{by equation (16).}\end{aligned} \tag{29}$$

Therefore, for any parameterization, the phase surfaces, Φ_0 can be reconstructed by integrating

$$\Phi_0(\vec{r}(\tau)) = \Phi_0(\vec{r}(\tau_0)) + \int_{\tau_0}^{\tau} \lambda(\vec{r}(\tau'))k^2\left(\vec{r}(\tau')\right)\, d\tau'. \tag{30}$$

We can simplify equation (30) if we choose

$$\lambda = k^{-2} \tag{31}$$

in which case

$$\frac{d\Phi_0}{d\tau} = 1 \quad \Longrightarrow \quad \Phi_0(\vec{r}(\tau)) = \Phi_0(\vec{r}(\tau_0)) + \tau - \tau_0 \tag{32}$$

which means that τ is equal to the phase up to an additive constant.

Furthermore, we have

$$\left|\frac{d\vec{r}}{d\tau}\right| = \lambda|\nabla\Phi_0| = \frac{1}{k} \tag{33}$$

for the magnitude of the tangent vector.

The way to think of this is that there are many rays side by side all parametrized by τ. If you integrate each of these rays forward in time from $\tau = \tau_0$ to $\tau = \tau_1$ then they will all go from one constant phase surface to another constant phase surface. This would not be true if you used any other parameterization (for example the arc length parameterization).

The ray equation is

$$k^2\frac{d}{d\tau}\left[k^2\frac{d\vec{r}}{d\tau}\right] = \frac{1}{2}\nabla\left(k^2\right). \tag{34}$$

2.3 Examples

It makes sense to use equation (23) and then determine Φ by integration. In the case of $k =$ constant it is clear that (23) corresponds to

$$\frac{d\vec{r}}{d\tau} = \text{constant} \tag{35}$$

and the rays are straight.

A more interesting example is to put ourselves in 2 dimensions and let the wave number only depend on height

$$k(\vec{x}) = k(z). \tag{36}$$

Now I'm going to change to $\vec{r}(\tau) = X(\tau)\hat{i} + Z(\tau)\hat{j}$ and write the ray equations (23) as

$$\begin{aligned} \frac{dX^2}{d\tau^2} &= 0 \\ \frac{d^2 Z}{d\tau^2} &= k k_z. \end{aligned} \tag{37}$$

These two second order equations need initial conditions which correspond to

$$X(0), \quad Z(0), \quad \frac{dX}{d\tau}(0), \quad \frac{dZ}{d\tau}(0). \tag{38}$$

Without loss of generality we can set the initial positions to zero - the initial derivatives are parameters of the problem.

Now let the initial wavenumber be $k = k_0$ and the initial derivatives be

$$\frac{dX}{d\tau}(0) = U_0 \quad \frac{dZ}{d\tau}(0) = V_0. \tag{39}$$

The magnitude equation (24)

$$\left(\frac{dX}{d\tau}\right)^2 + \left(\frac{dZ}{d\tau}\right)^2 = k^2 \tag{40}$$

implies that

$$U_0^2 + V_0^2 = k_0^2. \tag{41}$$

It is convenient to re-write the initial data in terms of magnitude and angle of the initial ray trajectory. Therefore

$$U_0 = k_0 \cos(\theta_0), \quad V_0 = k_0 \sin(\theta_0). \tag{42}$$

We can now easily solve the first equation in (37)

$$\frac{dX}{d\tau} = \text{constant} \equiv U_0 = k_0 \cos(\theta_0). \tag{43}$$

Substituting into the magnitude equation (40) and using the definition of k we find

$$\frac{dZ}{d\tau} = \pm\sqrt{\left(\frac{\omega}{c}\right)^2 - k_0^2 \cos(\theta_0)^2}. \tag{44}$$

A little more manipulation and (44) will reveal itself. The initial wavenumber is written in terms of the wave frequency and the initial wave speed as

$$k_0 = \frac{\omega}{c_0}. \tag{45}$$

It is customary to define the index of refraction as the ratio

$$n \equiv \frac{c_0}{c(\vec{r})}. \tag{46}$$

and substituting both of these definitions into (44) yields

$$\frac{dZ}{d\tau} = \pm k_0 \sqrt{n^2 - \cos(\theta_0)^2}. \tag{47}$$

Notice that at the initial point along the ray, $n = 1$ so that the right hand side is $\pm k_0 \sin(\theta_0)$. In general the radical on the right hand side can be written

$$\sin(\theta) = \sqrt{n^2 - \cos(\theta_0)^2} \tag{48}$$

Equation (48) tells us two things:

1. If c is a decreasing function of height then n is increasing and so is $\sin(\theta)$. Therefore a wave going into a medium of slower wave speed will turn so that its rays are pointing toward the direction of decreasing wave speed. This is the same point that I made about waves coming along shore where the rays become perpendicular (and the wave fronts become parallel) to the shore line.

2. If c is an increasing function of height then n is decreasing and so is $\sin(\theta)$. There can exist a height where $\sin(\theta) = 0$ i.e.

$$n(z_*) = \cos(\theta_0). \tag{49}$$

This is a turning height for the ray. The ray will turn at that height and then proceed downward. This is similar to the total internal reflection that we see in water/air.

The phase fronts of the wave can be reconstructed using the integral

$$\Phi_0(\vec{r}(\tau)) = \int_0^\tau k_0^2\, n^2(\vec{r}(\tau'))\, d\tau'. \tag{50}$$

3 The amplitude of the wave

The amplitude equation can, in fact, be recast as a conservation law. Begin with

$$\frac{2\nabla\Phi_1 \cdot \nabla\Phi_0}{k^2} = \nabla \cdot \left[\frac{i}{k^2}\nabla.\Phi_0\right] \tag{51}$$

Now, this transformation is far from obvious, but arises because we are trying to create an integrating factor for (51). Let

$$\Phi_1 = i\ln(G^{-\frac{1}{2}}) \quad \Longrightarrow \quad \nabla\Phi_1 = -\frac{i}{2}\frac{\nabla G}{G}. \tag{52}$$

Notice that if we substitute Φ_1 into the wave equation we find

$$F \propto \sqrt{G} \tag{53}$$

so that $\sqrt{G} \in \mathbf{R}$ describes the amplitude of the wave.

Substituting the definition (52) into the Amplitude Equation, we find

$$\nabla G \cdot \frac{\nabla \Phi_0}{k^2} + G\, \nabla \cdot \left[\frac{\nabla \Phi_0}{k^2}\right] = 0. \tag{54}$$

You can see that this is simply an application of product rule using the divergence. Therefore the amplitude equation becomes

$$\nabla \cdot \left[G \frac{\nabla \Phi_0}{k^2}\right] = 0. \tag{55}$$

Whenever one encounters a divergence of a vector field is equal to zero, the problem screams at you to use the divergence theorem. Imagine a volume in $D \subset \mathbf{R}^3$ which looks like a tube. The top and bottom of the tube are surfaces of constant phase, Φ_0 (of course the phase is different on the top than it is on the bottom). The sides of the tube consist of rays. The cross section does not have to be round. Integrating (55) over this volume and applying the divergence theorem we find

$$\int\int_{\partial D} G \frac{\nabla \Phi_0}{k^2} \cdot \hat{n}\; d^2A = 0 \tag{56}$$

where $\hat{n}$ is the outward normal to the surface which is the boundary of D, ∂D. Now the sides of the tube are parallel to the rays, meaning that their normals are perpendicular to surfaces of constant Φ_0. Therefore, the integral along the sides of the tube is equal to zero.

Let Σ_1 be the surface of constant phase which forms the end of the tube and Σ_0 be the surface of constant phase which forms the beginning of the tube. On either of these surfaces

$$\nabla \Phi_0 \parallel \hat{n} \quad \Longrightarrow \quad \nabla \Phi_0 \cdot \hat{n} = |\nabla \Phi_0| = k \tag{57}$$

where I have used the Eikonal equation for the last equality. Therefore equation (56) becomes

$$\int\int_{\Sigma_1} \frac{G}{k}\, d^2A = \int\int_{\Sigma_0} \frac{G}{k}\, d^2A \tag{58}$$

and replacing the wavenumber with the wave speed we find

$$\int\int_{\Sigma_1} c\, G\; d^2A = \int\int_{\Sigma_0} c\, G\; d^2A. \tag{59}$$

Equation (59) says that, if you integrate cG over a patch of a constant phase surface, then that is conserved along rays to the next patch of constant phase surface. For a small enough patch of surface, we can denote $c, , G$ as their average values over the patch, let A be the area of that patch and write

$$c\, GA = \text{constant along rays} \tag{60}$$

or alternatively

$$G = G_0\, \frac{c_0 A_0}{cA} \tag{61}$$

along rays, where the subscript 0 is taken along the initial surface.

As a simple example consider a constant wave speed, spherical symmetric wave front. Then the area of the spherical surface of constant Φ_0 is proportional to square of the distance from the origin

$$A \propto d^2 \quad \Longrightarrow \quad G \propto \frac{1}{d^2} \tag{62}$$

which further implies that the amplitudes of the wave

$$|F| \propto \sqrt{G} \propto \frac{1}{d}. \tag{63}$$

In this simple example, the amplitude of the wave is proportional to the inverse distance from the source. The energy density (which is proportional to the amplitude squared) is proportional to the inverse distance squared. This further implies that energy is conserved because is says that the area of the surface multiplied by the energy density on the surface is conserved as the wave propagates from one surface of constant phase to another.

The most general form of the conservation is provided in (59).

4 Two examples of $n(z)$

We will try two examples of $n(z)$, a decreasing and and increasing one

$$n(z) = \frac{L}{z}, \quad \text{and} \quad n(z) = \frac{z}{L}. \tag{64}$$

In both cases we integrate from the start point $(x, z) = (0, L)$ and plot the trajectories of the rays.

4.1 Increasing wave speed as a function of height.

We will use

$$n(z) = \frac{L}{z} \tag{65}$$

and solve the equations

$$\frac{dX}{d\tau} = k_0 \cos(\theta_0) \tag{66}$$

and

$$\frac{dZ}{d\tau} = \pm k_0 \sqrt{n^2 - \cos^2(\theta_0)} \tag{67}$$

with initial data $X(0) = 0$ and $Z(0) = L$. Therefore

$$X(\tau) = k_0 \cos(\theta_0)\, \tau \tag{68}$$

and we solve the Z equation by separation

$$\int_L^{Z(\tau)} \frac{dZ'}{k_0 \sqrt{\left(\frac{L}{Z'}\right)^2 - \cos^2(\theta_0)}} = \int_0^\tau d\tau'. \tag{69}$$

I chose the positive square root because we will begin with upward going waves. However, there will be a maximum and they will turn back. At the turning point we should shift to the negative square root. Let's solve this integral.

$$\begin{aligned}
\int_L^{Z(\tau)} \frac{dZ'}{k_0\sqrt{\left(\frac{L}{Z'}\right)^2 - \cos^2(\theta_0)}} &= \tau \\
\int_L^{Z(\tau)} \frac{Z'dZ'}{Lk_0\sqrt{1-\left(\frac{Z'}{L}\right)^2\cos^2(\theta_0)}} &= \tau \\
\frac{1}{2}\frac{L}{k_0}\int_1^{\left(\frac{Z}{L}\right)^2} \frac{du}{\sqrt{1-u\cos^2(\theta_0)}} &= \tau \quad \text{by substitution} \\
\frac{L}{k_0\cos^2(\theta_0)}\left[\sqrt{1-u\cos^2(\theta_0)}\right]_{\left(\frac{Z}{L}\right)^2}^{1} &= \tau \\
\sin(\theta_0) - \sqrt{1-\left(\frac{Z}{L}\right)^2\cos(\theta_0)^2} &= \frac{X\cos(\theta_0)}{L}
\end{aligned} \tag{70}$$

where I have used the solution for $X(\tau)$ to get an implicit equation for Z rather than a parametric equation. A quick manipulation and we arrive at

$$\left(\frac{L}{\cos(\theta_0)}\right)^2 = \left[X - L\tan(\theta_0)\right]^2 + Z^2. \tag{71}$$

This is a beautiful result, saying that rays which are emitted at angle θ_0 from the horizontal travel along circles of radius

$$\frac{L}{\cos(\theta_0)} \tag{72}$$

center on the point

$$(X_*, Z_*) = (L\tan(\theta_0), L). \tag{73}$$

Notice that the point $(X, Z) = (0, L)$ is the point of emission of the wave and lies along all of the circles.

Now we'd like to plot the constant phase curves

$$
\begin{aligned}
\Phi_0(\vec{r}(\tau)) &= \int_0^\tau k_0^2 \frac{L^2}{Z(\tau')^2}\, d\tau' \\
&= \int_0^\tau k_0^2 \frac{L^2}{Z(\tau')^2}\, d\tau' \\
&= \int_0^\tau k_0^2 \frac{1}{\frac{1}{\cos^2(\theta_0)} - \left[\frac{X(\tau')}{L} - \frac{\sin(\theta_0)}{\cos(\theta_0)}\right]^2}\, d\tau' \\
&= \int_0^\tau \frac{k_0^2 \cos(\theta_0)^2}{1 - \left[\frac{k_0 \cos^2(\theta_0)\tau'}{L} - \sin(\theta_0)\right]^2}\, d\tau' \\
&= \int_0^{\frac{k_0 \cos^2(\theta)}{L}\tau} \frac{k_0 L}{1 - [s - \sin(\theta_0)]^2}\, ds \\
&= \frac{k_0 L}{2} \int_0^{\frac{k_0 \cos^2(\theta)}{L}\tau} \frac{1}{1 + s - \sin(\theta_0)} + \frac{1}{1 - s + \sin(\theta_0)}\, ds \\
&= \frac{k_0 L}{2} \left[\ln(1 + s - \sin(\theta_0)) - \ln(1 - s + \sin(\theta_0))\right]\Big|_0^{\frac{k_0 \cos^2(\theta)}{L}\tau} \\
\Phi_0(\vec{r}(\tau)) &= \frac{k_0 L}{2} \left[\ln\left(\frac{1 + \frac{k_0 \cos^2(\theta)}{L}\tau - \sin(\theta_0)}{1 - \frac{k_0 \cos^2(\theta)}{L}\tau + \sin(\theta_0)}\right) - \ln\left(\frac{1 - \sin(\theta_0)}{1 + \sin(\theta_0)}\right)\right] \\
&= \frac{k_0 L}{2} \ln\left[\frac{1 - \sin(\theta_0) + \frac{X}{L}\cos(\theta_0)}{1 + \sin(\theta_0) - \frac{X}{L}\cos(\theta_0)} \quad \frac{1 + \sin(\theta_0)}{1 - \sin(\theta_0)}\right] \\
&= \frac{k_0 L}{2} \ln\left[\frac{1 - \sin^2(\theta_0) + \frac{X}{L}\cos(\theta_0)\left[1 + \sin(\theta_0)\right]}{1 - \sin^2(\theta_0) - \frac{X}{L}\cos(\theta_0)\left[1 - \sin(\theta_0)\right]}\right] \\
&= \frac{k_0 L}{2} \ln\left[\frac{\cos(\theta_0) + \frac{X}{L}\left[1 + \sin(\theta_0)\right]}{\cos(\theta_0) - \frac{X}{L}\left[1 - \sin(\theta_0)\right]}\right] \\
&= \frac{k_0 L}{2} \ln\left[\frac{1 + \frac{X}{L}\left[\sec(\theta_0) + \tan(\theta_0)\right]}{1 - \frac{X}{L}\left[\sec(\theta_0) - \tan(\theta_0)\right]}\right]
\end{aligned} \tag{74}
$$

What we really want is an expression of Φ_0 in terms of X and Z. To do this we have eliminated τ in favor of X and and must do that for $\cos(\theta_0)$ in favor of X, Z. In order to do this, let's look at the ray curves

$$
\begin{aligned}
\frac{1}{\cos^2(\theta_0)} &= \left(\frac{X}{L}\right)^2 - 2\tan(\theta_0)\frac{X}{L} + \tan^2(\theta_0) + \left(\frac{Z}{L}\right)^2 \\
2\tan(\theta_0)\frac{X}{L} &= \left(\frac{X}{L}\right)^2 + \left(\frac{Z}{L}\right)^2 - 1 \\
\tan(\theta_0) &= \frac{X^2 + Z^2 - L^2}{2XL}.
\end{aligned} \tag{75}
$$

We can think of the Opposite side of a triangle as

$$\mathrm{O} = X^2 + Z^2 - L^2 \tag{76}$$

and the adjacent side as

$$\mathrm{A} = 2XL \tag{77}$$

so that the hypotenuse is

$$\mathrm{H}^2 = (X^2 + Z^2 - L^2)^2 + (2XL)^2 \tag{78}$$

and the secant is

$$\sec(\theta_0) = \frac{\mathrm{H}}{\mathrm{A}} = \frac{\sqrt{(X^2 + Z^2 - L^2)^2 + (2XL)^2}}{2XL} \tag{79}$$

Continuing to simplify the phase

$$\begin{aligned}
e^{\frac{2\Phi_0}{k_0 L}} &= \frac{2L^2 + 2XL\ [\sec(\theta_0) + \tan(\theta_0)]}{2L^2 - 2XL\ [\sec(\theta_0) - \tan(\theta_0)]} \\
&= \frac{2L^2 + X^2 + Z^2 - L^2 + 2XL\sec(\theta_0)}{2L^2 + X^2 + Z^2 - L^2 - 2XL\sec(\theta_0)} \\
&= \frac{L^2 + X^2 + Z^2 + 2XL\sec(\theta_0)}{L^2 + X^2 + Z^2 - 2XL\sec(\theta_0)} \\
\Gamma &= \frac{L^2 + X^2 + Z^2 + \sqrt{(X^2 + Z^2 - L^2)^2 + (2XL)^2}}{L^2 + X^2 + Z^2 - \sqrt{(X^2 + Z^2 - L^2)^2 + (2XL)^2}} \\
(\Gamma + 1)\sqrt{(X^2 + Z^2 - L^2)^2 + (2XL)^2} &= (\Gamma - 1)\left[L^2 + X^2 + Z^2\right] \\
(\Gamma + 1)^2\left[(X^2 + Z^2 + L^2)^2 - (2ZL)^2\right] &= (\Gamma - 1)^2\left[L^2 + X^2 + Z^2\right]^2 \\
4\Gamma\left(X^2 + Z^2 + L^2\right)^2 &= (\Gamma + 1)^2\,(2ZL)^2 \\
X^2 + Z^2 + L^2 &= \frac{\Gamma + 1}{\sqrt{\Gamma}}\ ZL \\
X^2 + Z^2 - \frac{\Gamma + 1}{\sqrt{\Gamma}}\ ZL + \frac{(\Gamma + 1)^2}{4\Gamma}L^2 + L^2 &= \frac{(\Gamma + 1)^2}{4\Gamma}L^2 \\
X^2 + \left[Z - \left(\frac{\Gamma + 1}{2\sqrt{\Gamma}}\right)L\right]^2 &= \frac{(\Gamma - 1)^2}{4\Gamma}\ L^2 \\
X^2 + \left[Z - \left(\frac{\Gamma^{\frac{1}{2}} + \Gamma^{-\frac{1}{2}}}{2}\right)L\right]^2 &= \left(\frac{\Gamma^{\frac{1}{2}} - \Gamma^{-\frac{1}{2}}}{2}\right)^2\ L^2 \\
X^2 + \left[Z - L\cosh\left(\frac{\Phi_0}{k_0 L}\right)\right]^2 &= L^2\sinh^2\left(\frac{\Phi_0}{k_0 L}\right)
\end{aligned} \tag{80}$$

Amazingly, these curves are circles also - of different radii and centers depending on the value of Φ_0. I have to admit I was completely shocked by this result, and after I plotted it (below) I had to work it out analytically to be sure. Notice when $\Phi_0 \to 0$ then $\Gamma \to 1$ and the radius $\to 0$ while the center is at $(X_c, Z_c) \to (0, L)$. The constant phase curves and constant rays are plotted in figure (1). [Another interesting thing to note is that the bottom points of all the phase circles accumulates at $Z = 0$ in the limit of large Φ_0.

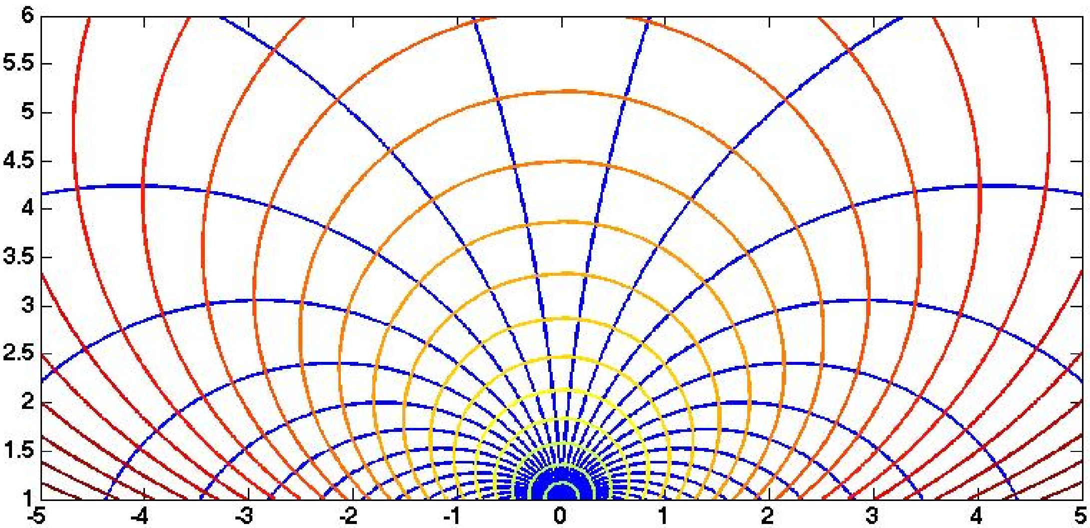

Figure 1: Rays (blue) and phase lines (yellow to red) for the case of wave speed increasing with height. X and Z are measured in units of L.

4.2 Decreasing wave speed as a function of height.

In this case we use

$$n(z) = \left(\frac{z}{L}\right)^{\frac{1}{2}} \tag{81}$$

and solve the same equations as before. The integral for Z now becomes

$$\begin{aligned}
\int_L^{Z(\tau)} \frac{dZ'}{k_0\sqrt{\frac{Z'}{L} - \cos^2(\theta_0)}} &= \int_0^\tau d\tau' \\
\int_1^{\frac{Z(\tau)}{L}} \frac{ds}{\sqrt{s - \cos^2(\theta_0)}} &= \frac{k_0}{L}\int_0^\tau d\tau' \\
\sqrt{\frac{Z}{L} - \cos^2(\theta_0)} - \sin(\theta_0) &= 2\frac{k_0}{L}\tau \\
Z &= L\cos^2(\theta_0) + \frac{(2k_0\tau + L\sin(\theta_0))^2}{L}
\end{aligned} \tag{82}$$

and the result for X is the same as before

$$X = k_0\cos(\theta_0)\tau. \tag{83}$$

So we can see that the rays are parabolas

$$\begin{aligned}
Z &= L\cos^2(\theta_0) + \frac{1}{L}\left(\frac{2X}{\cos(\theta_0)} + L\sin(\theta_0)\right)^2 \\
&= L\cos^2(\theta_0) + \frac{4}{L\cos^2(\theta_0)}\left(X + \frac{\sin(\theta_0)\cos(\theta_0)}{2}L\right)^2,
\end{aligned} \tag{84}$$

whose vertices are at

$$Z_V = L\cos^2(\theta_0), \quad X_v = -\frac{\sin(\theta_0)\cos(\theta_0)}{2}L \tag{85}$$

and whose curvature is

$$\frac{4}{L\cos^2(\theta_0)}. \tag{86}$$

Constant phase curves are attained by integrating

$$\begin{aligned} \Phi_0(\vec{r}(\tau)) &= \int_0^\tau k_0^2 \frac{Z(\tau')}{L}\, d\tau' \\ &= \frac{k_0^2}{L^2}\int_0^\tau \left[4k_0^2\tau'^2 + 4k_0 L\sin(\theta_0)\tau' + L^2\right]\, d\tau' \\ &= k_0^2\left[\frac{4k_0^2\tau^3}{3L^2} + \frac{2k_0\sin(\theta_0)\tau^2}{L} + \tau\right] \\ &= k_0^2\tau\left[\frac{4k_0^2\tau^2}{3L^2} + \frac{2k_0\sin(\theta_0)\tau}{L} + 1\right]. \end{aligned} \tag{87}$$

We would like to get Φ_0 in terms of (X, Z) only - i.e try to eliminate τ, Φ_0. Take the Z and Z equations

$$\begin{aligned} LZ &= 4k_0^2\tau^2 + 4k_0 L\sin(\theta_0)\tau + L^2 \\ X &= k_0\cos(\theta_0)\tau. \end{aligned} \tag{88}$$

We can use the second equation to write

$$k_0\tau\sin(\theta_0) = \sqrt{k_0^2\tau^2 - X^2} \tag{89}$$

and substitute into the Z equation

$$\begin{aligned} \left(LZ - L^2 - 4k_0^2\tau^2\right)^2 &= 16L^2\left(k_0^2\tau^2 - X^2\right) \\ \left(LZ - L^2 - 4s\right)^2 &= 16L^2\left(s - X^2\right) \quad \text{let } s = (k_0\tau)^2 \text{ for now} \\ L^2Z^2 + L^4 + 16s^2 - 2L^3Z + 8s(L^2 - LZ) &= 16L^2 s - 16L^2X^2 \\ 16s^2 - 8s(L^2 + LZ) &= -16L^2X^2 - (L^2Z^2 + L^4 - 2L^3Z) \\ 16s^2 - 8s(L^2 + LZ) + (L^2 + LZ)^2 &= -16L^2X^2 - (L^2Z^2 + L^4 - 2L^3Z) + (L^2 + LZ)^2 \\ \left[4s - \left(L^2 + LZ\right)\right]^2 &= 4L^3Z - 16L^2X^2 \\ 4s &= \left(L^2 + LZ\right) \pm 2L\sqrt{LZ - (2X)^2} \end{aligned} \tag{90}$$

Since at $X = 0$, $Z = L$ we want $\tau = 0$, then we choose the negative square sign for the square root

$$2k_0\tau = \sqrt{(L^2 + LZ) - 2L\sqrt{LZ - (2X)^2}}, \tag{91}$$

and replace $\sin(\theta_0)$ by

$$k_0 \tau \sin(\theta_0) = \left[\frac{(L^2 + LZ) - 2L\sqrt{LZ - (2X)^2}}{4} - X^2 \right]^{\frac{1}{2}} . \qquad (92)$$

Both of these terms must be substituted into the equation for Φ_0 to get the expression for Φ_0 as a function of (X, Z).

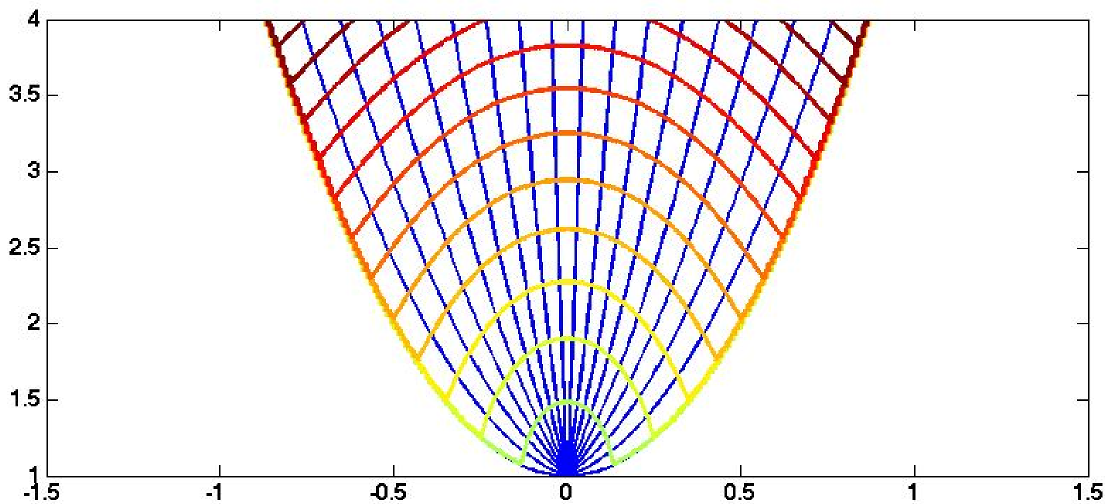

Figure 2: Rays (blue) and phase lines (yellow to red) for the case of wave speed decreasing with height. X and Z are measured in units of L.

In figure 2 we see the rays (blue) and the wave fronts (green to red) from a collection of rays emanating with angles $0 \leq \theta_0 \leq \pi$. The flattening of wave fronts is evident already.

In figure 3 I set a collection of rays out at all angles. The ones at angles $\pi < \theta_0 < 2\pi$ turn around and come up again - thereby crossing the other rays. The wave fronts are shown also - and you see two envelopes. The first envelope is $Z = 4X^2 + 1$ which is the same for both examples. It is the envelope of rays initially pointing upward. The second is $Z = 4X^2$ which also contains the rays initially pointing downward.

Although the phases bunch up at the perimeter of the outer envelope, the amplitudes would decrease (see the calculation I did before regarding the amplitudes). I am also pretty sure that neither the phase nor the amplitude is unique if rays cross - which would be between the two envelopes.

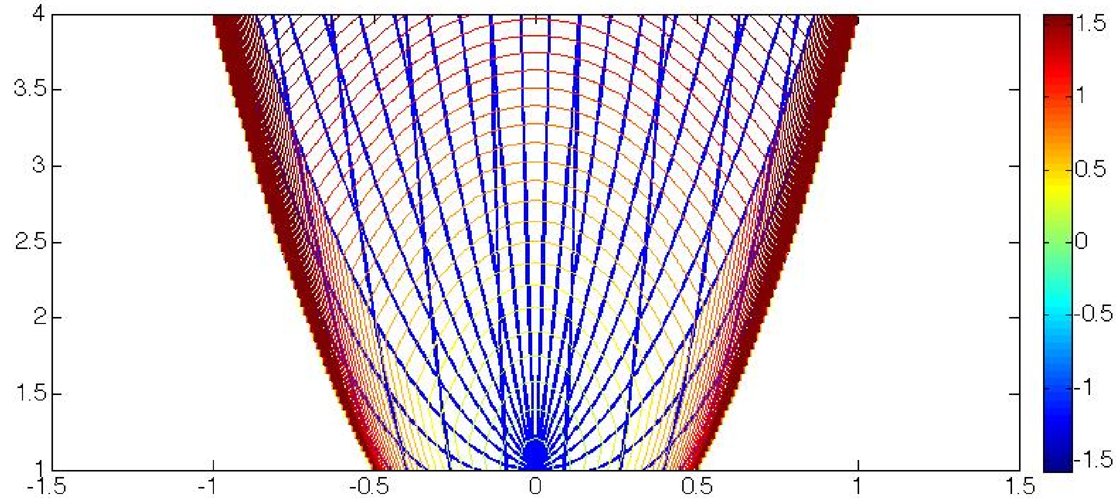

Figure 3: Rays (blue) and phase lines (yellow to red) for the case of wave speed decreasing with height. X and Z are measured in units of L.

4.3 A different decreasing wave speed as a function of height.

In this case we use

$$n(z) = \frac{z}{L} \tag{93}$$

and solve the same equations as before. The integral for Z now becomes

$$\begin{aligned}
\int_L^{Z(\tau)} \frac{dZ'}{k_0\sqrt{\left(\frac{Z'}{L}\right)^2 - \cos^2(\theta_0)}} &= \int_0^\tau d\tau' \\
\int_1^{\frac{Z(\tau)}{L}} \frac{ds}{\sqrt{s^2 - \cos^2(\theta_0)}} &= \frac{k_0}{L}\int_0^\tau d\tau' \\
\ln\left[\sqrt{s^2 - \cos^2(\theta_0)} + s\right]\Big|_1^{\frac{Z(\tau)}{L}} &= \frac{k_0}{L}\tau \\
\ln\left[\frac{\sqrt{Z^2 - L^2\cos^2(\theta_0)} + Z}{L\left(1 + \sin(\theta_0)\right)}\right] &= \frac{X}{L\cos(\theta_0)}.
\end{aligned} \tag{94}$$

Just to make this easier to work with, define

$$x = \frac{X}{L\cos(\theta_0)}, \quad z = \frac{Z}{L\cos(\theta_0)}, \quad a = \frac{1 + \sin(\theta_0)}{\cos(\theta_0)}, \tag{95}$$

to find the scaled implicit equation for rays

$$
\begin{aligned}
\sqrt{z^2-1}+z &= ae^x \\
z^2-1 &= (ae^x-z)^2 \\
0 &= a^2e^{2x}-2ae^xz+1 \\
2z &= ae^x+a^{-1}e^{-x} \\
z &= \cosh\left(x+\ln(a)\right).
\end{aligned}
\tag{96}
$$

Rewriting in terms of X, Z yields

$$
Z = L\cos(\theta_0)\ \cosh\left[\frac{X}{L\cos(\theta_0)}+\ln\left(\frac{1+\sin(\theta_0)}{\cos(\theta_0)}\right)\right]. \tag{97}
$$

These solutions imply that even waves edited horizontally, $\theta_0 = 0$, will eventually turn upward - the envelope of the waves is

$$
Z_{\text{envelope}} = L\cosh\left[\frac{X}{L}\right]. \tag{98}
$$

Again we know that surfaces of constant Φ_0 are given by

$$
\begin{aligned}
\Phi_0(\vec{r}(\tau)) &= \int_0^\tau k_0^2\frac{Z(\tau')^2}{L^2}\ d\tau' \\
&= k_0^2\cos(\theta_0)\int_0^\tau \cosh\left[\frac{k_0\tau'}{L}+\ln\left(\frac{1+\sin(\theta_0)}{\cos(\theta_0)}\right)\right]\ d\tau' \\
&= k_0L\cos(\theta_0)\sinh\left[\frac{k_0\tau}{L}+\ln\left(\frac{1+\sin(\theta_0)}{\cos(\theta_0)}\right)\right] \\
&= k_0L\cos(\theta_0)\sinh\left[\frac{X}{L\cos(\theta_0)}+\ln\left(\frac{1+\sin(\theta_0)}{\cos(\theta_0)}\right)\right] \\
&= k_0L\cos(\theta_0)\sinh\left\{\cosh^{-1}\left[\frac{Z}{L\cos(\theta_0)}\right]\right\}
\end{aligned}
\tag{99}
$$

Again we could try to eliminate θ_0 and τ in favor of X, Z - but that seems nearly impossible.