Consider the forced simple harmonic oscillator with homogeneous initial conditions

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x = \cos(\Omega t), \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 0. \tag{1}$$

Solve this for arbitrary parameters, Γ, ω, Ω. Then take the resonant limit where $\Omega \to \omega$.

Homogeneous solution

The linear operator

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega^2 x = 0 \tag{2}$$

corresponds to damped harmonic motion so we can guess a functional form

$$x_H(t) = Ae^{-\Gamma t}\sin(\lambda t) + Be^{-\Gamma t}\cos(\lambda t) \tag{3}$$

where λ and Γ are yet unknown. The derivative of the homogeneous solution is

$$\frac{dx_H}{dt} = -\Gamma\left[Ae^{-\Gamma t}\sin(\lambda t) + Be^{-\Gamma t}\cos(\lambda t)\right] + \lambda\left[Ae^{-\Gamma t}\cos(\lambda t) - Be^{-\Gamma t}\sin(\lambda t)\right] \tag{4}$$

and the second derivative is

$$\begin{aligned}\frac{d^2x_H}{dt^2} &= \Gamma^2\left[Ae^{-\Gamma t}\sin(\lambda t) + Be^{-\Gamma t}\cos(\lambda t)\right] \\ &\quad - 2\Gamma\lambda\left[Ae^{-\Gamma t}\cos(\lambda t) - Be^{-\Gamma t}\sin(\lambda t)\right] \\ &\quad - \lambda^2\left[Ae^{-\Gamma t}\sin(\lambda t) + Be^{-\Gamma t}\cos(\lambda t)\right] \\ &= \left(\Gamma^2 - \lambda^2\right)\left[Ae^{-\Gamma t}\sin(\lambda t) + Be^{-\Gamma t}\cos(\lambda t)\right] - 2\Gamma\lambda\left[Ae^{-\Gamma t}\cos(\lambda t) - Be^{-\Gamma t}\sin(\lambda t)\right].\end{aligned} \tag{5}$$

Substituting these derivatives into the homogeneous equation yields

$$\begin{aligned}&\left(\Gamma^2 - \lambda^2 - \Gamma\gamma + \omega^2\right)\left[Ae^{-\Gamma t}\sin(\lambda t) + Be^{-\Gamma t}\cos(\lambda t)\right] \\ &\qquad + (\gamma\lambda - 2\Gamma\lambda)\left[Ae^{-\Gamma t}\cos(\lambda t) - Be^{-\Gamma t}\sin(\lambda t)\right] = 0.\end{aligned} \tag{6}$$

If $x_H(t)$ is to be a solution for all time, then equation (6) must be true for all time meaning that

$$\Gamma = \frac{\gamma}{2}, \quad \lambda = \sqrt{\omega^2 - \frac{\gamma^2}{4}}. \tag{7}$$

The decay rate of the homogeneous solution is $\gamma/2$. The frequency of oscillation of the homogeneous solution is λ; notice that this frequency is equal to the natural frequency of the oscillator, ω, modified by the damping rate, γ. If $\gamma = 2\omega$ then the frequency is equal to zero - this is called a critically damped oscillator. If $\gamma > 2\omega$ then the frequency is imaginary; this just corresponds to stronger decaying solutions and is called over damped. If $\gamma < 2\omega$, the solutions oscillate and decay; this is called underdamped.

Let us focus on the underdamped case - this is the case you usually see when you force a spring yourself. The homogeneous solution is

$$x_H(t) = e^{-\frac{\gamma t}{2}} \left[A \sin(\lambda t) + B \cos(\lambda t) \right] \tag{8}$$

with λ given from equation (7).

Now we calculate the particular solution associated with the right hand side forcing. There are many ways to do this (Laplace transforms and the like) but one simple way is to realize that this is a forced spring and it is being forced at the frequency Ω. One thing is to try a particular solution of the form

$$x_P(t) = C \sin(\Omega t) + D \cos(\Omega t), \tag{9}$$

whose first derivative is

$$\frac{dx_P}{dt}(t) = \Omega \left[C \cos(\Omega t) - D \sin(\Omega t) \right], \tag{10}$$

and whose second derivative is

$$\frac{d^2 x_P}{dt^2}(t) = -\Omega^2 \left[C \sin(\Omega t) + D \cos(\Omega t) \right]. \tag{11}$$

Substituting into (1) yields

$$\left(\omega^2 - \Omega^2\right) \left[C \sin(\Omega t) + D \cos(\Omega t) \right] + \gamma\Omega \left[C \cos(\Omega t) - D \sin(\Omega t) \right] = \cos(\Omega t) \tag{12}$$

and collecting coefficients of sine and cosine yields

$$\left[\left(\omega^2 - \Omega^2\right) C - \gamma\Omega D \right] \sin(\Omega t) + \left[\left(\omega^2 - \Omega^2\right) D + \gamma\Omega C - 1 \right] \cos(\Omega t) = 0. \tag{13}$$

Again, this equation must equal zero for all time. Therefore we have to solve 2 linear algebraic equations for C and D

$$\begin{aligned} \left(\omega^2 - \Omega^2\right) C - \gamma\Omega D &= 0 \\ \left(\omega^2 - \Omega^2\right) D + \gamma\Omega C &= 1. \end{aligned} \tag{14}$$

You can think of this as a 2x2 matrix acting on the column vector $[C, D]^T$ or you can just solve it by elimination (which I will do). Eliminating D yields

$$C = \frac{\gamma\Omega}{\left[(\omega^2 - \Omega^2)^2 + (\gamma\Omega)^2\right]} \tag{15}$$

and then solving for D gives

$$D = \frac{(\omega^2 - \Omega^2)}{\left[(\omega^2 - \Omega^2)^2 + (\gamma\Omega)^2\right]}. \tag{16}$$

So the particular solution is

$$x_P(t) = \frac{\gamma\Omega\sin(\Omega t) + (\omega^2 - \Omega^2)\cos(\Omega t)}{\left[(\omega^2 - \Omega^2)^2 + (\gamma\Omega)^2\right]}. \tag{17}$$

Since x_P is a sum of sine and cosine with the same frequency, Ω we can write it as a phase shifted cosine (or sine) as

$$x_P(t) = \frac{\cos(\Omega t - \Phi)}{\left[(\omega^2 - \Omega^2)^2 + (\gamma\Omega)^2\right]^{\frac{1}{2}}}. \tag{18}$$

where the phase shift is given by

$$\tan(\Phi) = \frac{\gamma\Omega}{\omega^2 - \Omega^2}. \tag{19}$$

Now the solution to the initial value problem is a linear combination of the homogeneous and particular solutions

$$x(t) = Ae^{-\gamma t/2}\sin(\lambda t) + Be^{-\gamma t/2}\cos(\lambda t) + \frac{\cos(\Omega t - \Phi)}{\left[(\omega^2 - \Omega^2)^2 + (\gamma\Omega)^2\right]^{\frac{1}{2}}} \tag{20}$$

where A and B must be determined by initial data, λ is given by (7) and Φ by (19). Now $x(0) = 0$ means

$$B = -\frac{\cos(\Phi)}{\left[(\omega^2 - \Omega^2)^2 + (\gamma\Omega)^2\right]^{\frac{1}{2}}}, \tag{21}$$

while $\frac{dx}{dt}(0) = 0$ means

$$0 = -\frac{\gamma}{2}B + \lambda A - \frac{\Omega\sin(\Phi)}{\left[\left(\omega^2-\Omega^2\right)^2+\left(\gamma\Omega\right)^2\right]^{\frac{1}{2}}} \tag{22}$$

so that

$$\lambda A = \frac{\Omega\sin(\Phi) - \frac{\gamma}{2}\cos(\Phi)}{\left[\left(\omega^2-\Omega^2\right)^2+\left(\gamma\Omega\right)^2\right]^{\frac{1}{2}}}. \tag{23}$$

Putting this all together may seem daunting, but it is worth it when we plot the solution. The solution is

$$x(t) = \frac{1}{\left[\left(\omega^2-\Omega^2\right)^2+\left(\gamma\Omega\right)^2\right]^{\frac{1}{2}}} \left\{e^{-\frac{\gamma t}{2}}\left[\left(\frac{\Omega\sin(\Phi)-\frac{\gamma}{2}\cos(\Phi)}{\lambda}\right)\sin(\lambda t) - \cos(\Phi)\cos(\lambda t)\right] + \cos(\Omega t - \Phi)\right\} \tag{24}$$

The terms in (24) which decay exponentially are called the transient. These correspond to an exponentially decaying and oscillating solution. After they die out, the oscillator goes to the particular solution

$$x(t) \longrightarrow x_P(t) = \frac{\cos(\Omega t - \Phi)}{\left[\left(\omega^2-\Omega^2\right)^2+\left(\gamma\Omega\right)^2\right]^{\frac{1}{2}}} \tag{25}$$

which is just a cosine in time. The result says that *eventually* the solution will oscillate at the forcing frequency, Ω but will not be in phase with the force since $\Phi \neq 0$. The solution will either chase the force or lead the force, depending on whether or not $\omega < \Omega$ or $\omega > \Omega$.

There is another term to consider, the *response amplitude*

$$\mathcal{A}(\Omega) = \frac{1}{\left[\left(\omega^2-\Omega^2\right)^2+\left(\gamma\Omega\right)^2\right]^{\frac{1}{2}}} \tag{26}$$

which you can consider to be a function of the forcing frequency. For small damping, $\mathcal{A}(\Omega)$ is a sharply peaked function of Ω. This means that if the forcing is *on resonance* (i.e. $\Omega = \omega$) then there is a strong responce by the oscillator. Some response amplitude curves are plotted below.

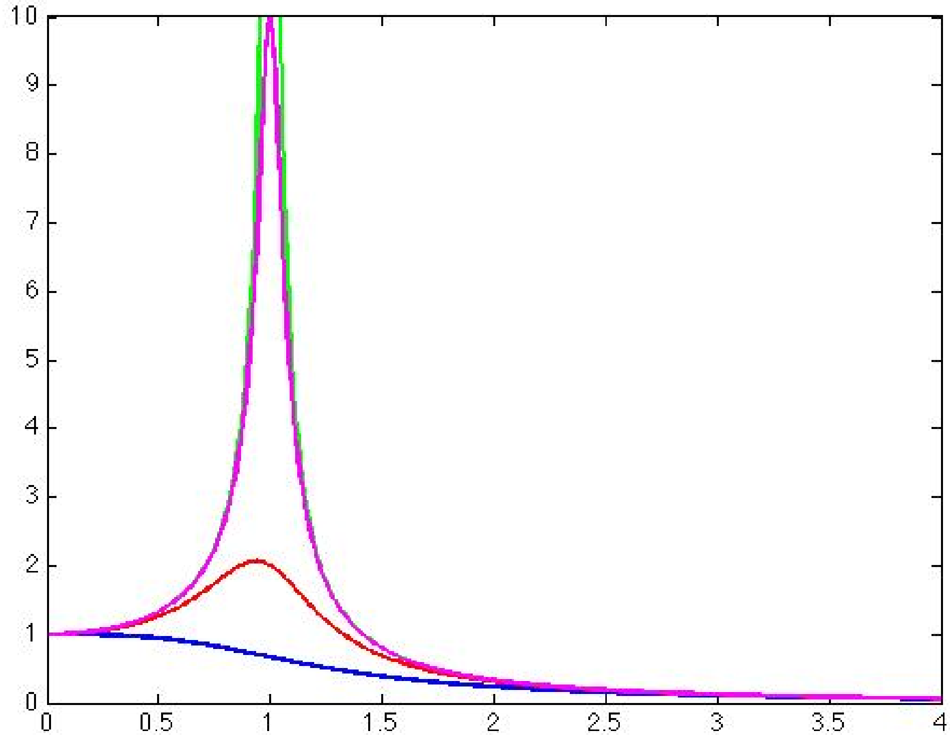

Figure 1: For example resonance response curves, $\mathcal{A}(\Omega)$ using $\omega = 1$ and $\gamma = 1.5, .5, .1, .05$. The smaller the value of γ, the sharper the peak at the resonance, $\Omega = 1$.