1 Pointwise convergence of Fourier series

This is the proof of pointwise convergence of the Fourier series of C^1 functions, $f(x)$ on the interval $-L \leq x < L$. The theorem is that, for such functions,

$$\lim_{N\to\infty} |\tilde{f}(x) - S_N(x)| \to 0 \tag{1}$$

for all x in the interval. $S_N(x)$ is the partial sum of the Fourier Series, i.e.

$$FS\ f(x) = \lim_{N\to\infty} S_N(x) \tag{2}$$

The adjusted function $\tilde{f}(x) = f(x)$ at all points where $f(x)$ is continuous and at points of discontinuity, x_k, it equals the average of the values of $f(x)$ on either side of the discontinuity

$$\tilde{f}(x_k) = \frac{f(x_k^+) + f(x_k^-)}{2}. \tag{3}$$

Outside of the interval, $f(x)$ is its periodized counterpart.

Proof:
We begin by writing out the Fourier series WITH the expression for the coefficients

$$\begin{aligned}
S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \frac{1}{L}\int_{-L}^{L}\sum_{n=1}^{N}\left[f(y)\cos\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + f(y)\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right] dy \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \frac{1}{L}\int_{-L}^{L}\sum_{n=1}^{N} f(y)\ \cos\left(\frac{n\pi\,(x-y)}{L}\right)\ dy \\
&= \frac{1}{L}\int_{-L}^{L} f(y)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos\left(\frac{n\pi\,(x-y)}{L}\right)\right] dy.
\end{aligned} \tag{4}$$

This motivates the definition of the function

$$D_N(z) = \frac{1}{2} + \sum_{n=1}^{N}\cos\left(\frac{n\pi z}{L}\right) \tag{5}$$

which is called the Dirichlet kernel. Let's look at the related series

$$
\begin{aligned}
\frac{1}{2}+\sum_{n=1}^{N}\cos(n\theta) &= \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{N}e^{in\theta}+e^{-in\theta} \\
&= \frac{1}{2}\sum_{n=-N} Ne^{in\theta} \\
&= \frac{e^{-iN\theta}}{2} \quad \sum_{n=0}^{2N}e^{in\theta} \\
&= \frac{\xi^{-N}}{2} \quad \sum_{n=0}^{2N}\xi^{n} \quad \text{where} \quad \xi\equiv e^{i\theta}.
\end{aligned} \tag{6}
$$

The last series in (6) is the geometric series

$$\sigma_M = 1+\xi+\xi^2+\xi^3+...+\xi^M \tag{7}$$

and notice the two relations

$$\sigma_{M+1}=\sigma_M+\xi^{M+1}, \quad \text{and} \quad \sigma_{M+1}=1+\xi\,\sigma_M. \tag{8}$$

Eliminating σ_{M+1} we find that the sum of the geometric series is

$$\sigma_M=\frac{1-\xi^{M+1}}{1-\xi}. \tag{9}$$

Substituting this into the cosine series we find

$$
\begin{aligned}
\frac{1}{2}+\sum_{n=1}^{N}\cos(n\theta) &= \frac{\xi^{-N}}{2}\,\frac{1-\xi^{2N+1}}{1-\xi} \\
&= \frac{\xi^{-N}}{2}\,\frac{\xi^{N+\frac{1}{2}}}{\xi^{\frac{1}{2}}}\,\frac{\xi^{N+\frac{1}{2}}-\xi^{-N-\frac{1}{2}}}{\xi^{\frac{1}{2}}-\xi^{-\frac{1}{2}}} \\
&= \frac{1}{2}\,\frac{\xi^{N+\frac{1}{2}}-\xi^{-N-\frac{1}{2}}}{\xi^{\frac{1}{2}}-\xi^{-\frac{1}{2}}} \\
&= \frac{1}{2}\,\frac{e^{i\theta\left(N+\frac{1}{2}\right)}-e^{-i\theta\left(N+\frac{1}{2}\right)}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}} \\
&= \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\left[\frac{\theta}{2}\right]}
\end{aligned} \tag{10}
$$

which is the form that we will need to analyze the Dirichlet Kernel.

Using the series form of the Dirichlet Kernel, the partial sum of the Fourier series is written as

$$
\begin{aligned}
S_N(x) &= \frac{1}{L}\int_{-L}^{L} f(y) D_N(x-y)\, dy \\
&= \frac{1}{L}\int_{-L-x}^{L-x} f(z+x) D_N(-z)\, dz \\
&= \frac{1}{L}\int_{-L-x}^{L-x} f(z+x) D_N(z)\, dz \quad \text{by symmetry of } D_n(z) \\
&= \frac{1}{L}\int_{-L}^{L} f(z+x) D_N(z)\, dz + \frac{1}{L}\int_{-L-x}^{-L} f(z+x) D_N(z)\, dz + \frac{1}{L}\int_{L}^{L-x} f(z+x) D_N(z)\, dz
\end{aligned}
\tag{11}
$$

For the last two integrals let $p(z) = f(z+x)D_N(z)$, which is a periodic function of x with period $2L$. These last two integrals become

$$
\begin{aligned}
&= \frac{1}{L}\int_{-L-x}^{-L} p(z)\, dz + \frac{1}{L}\int_{L}^{L-x} p(z)\, dz \\
&= \frac{1}{L}\int_{-L-x}^{-L} p(z+2L)\, dz + \frac{1}{L}\int_{L}^{L-x} p(z)\, dz \\
&= \frac{1}{L}\int_{L-x}^{L} p(u)\, du + \frac{1}{L}\int_{L}^{L-x} p(z)\, dz \\
&= 0,
\end{aligned}
\tag{12}
$$

so that the partial sum simplifies to

$$
S_N(x) = \frac{1}{L}\int_{-L}^{L} f(z+x) D_N(z)\, dz. \tag{13}
$$

A study of $S_N(x)$ has been reduced to a study of $D_N(z)$. First note the integral

$$
\frac{1}{L}\int_{-L}^{L} D_N(z)\, dz = 1 \quad \forall \quad N \tag{14}
$$

since the integral of all the cosine terms vanish. Multiplying both sides of this expression by $f(x)$ yields

$$
f(x) = \frac{1}{L}\int_{-L}^{L} f(x)\, D_N(z)\, dz \quad \forall \quad N. \tag{15}
$$

Consider the difference between the adjusted function and the partial sum of the Fourier Series and substitute the rational function form of the Dirichlet Kernel

$$
\begin{aligned}
\tilde{f}(x) - S_N(x) &= \frac{1}{L}\int_{-L}^{L} \left[\tilde{f}(x) - f(x+z)\right] D_N(z)\, dz \\
&= \frac{1}{L}\int_{-L}^{L} \left[\tilde{f}(x) - f(x+z)\right] \frac{\sin\left[\left(N+\frac{1}{2}\right)\frac{\pi z}{L}\right]}{2\sin\left[\frac{\pi z}{2L}\right]}\, dz \\
&= \frac{1}{2L}\int_{-L}^{L} \left[\tilde{f}(x) - f(x+z)\right] \left[\sin\left(\frac{N\pi z}{L}\right)\cot\left(\frac{\pi z}{2L}\right) + \cos\left(\frac{N\pi z}{L}\right)\right]\, dz
\end{aligned}
\tag{16}
$$

Finally, we need the Riemann-Lebesgue Lemma. Let $f(x)$ be a piecewise C^1 function on the interval. Then

$$\begin{aligned}
\lim_{N\to\infty}\left|\frac{1}{L}\int_{-L}^{L} f(z)e^{i\frac{N\pi z}{L}}\,dz\right| &= \lim_{N\to\infty}\left|\frac{1}{iN\pi}\left[f(L)e^{iN\pi} - f(-L)e^{-iN\pi} + \int_{-L}^{L} f'(z)e^{i\frac{N\pi z}{L}}\,dz\right]\right| \\
&\leq \lim_{N\to\infty}\frac{1}{iN\pi}\left[|f(L)|\,|e^{iN\pi}| + |f(-L)|\,\left|e^{-iN\pi}\right| + \left|\int_{-L}^{L} f'(z)e^{i\frac{N\pi z}{L}}\,dz\right|\right] \\
&\leq \lim_{N\to\infty}\frac{1}{iN\pi}\left[|f(L)| + |f(-L)| + \int_{-L}^{L}|f'(z)|\,dz\right] \\
&\leq \lim_{N\to\infty}\frac{1}{iN\pi}\left[|f(L)| + |f(-L)| + 2LM\right]
\end{aligned} \tag{17}$$

where M is the maximum of the first derivative on the interval. Integration by parts is allowed for piecewise C^1 functions, $f(x)$. Clearly this expression goes to zero in the limit.

Now returning to the expression for the partial sum and the adjusted function

$$\begin{aligned}
\lim_{N\to\infty}\left|\tilde{f}(x) - S_N(x)\right| &= \lim_{N\to\infty}\left|\frac{1}{2L}\int_{-L}^{L}\left[\tilde{f}(x) - f(x+z)\right]\left[\sin\left(\frac{N\pi z}{L}\right)\cot\left(\frac{\pi z}{2L}\right) + \cos\left(\frac{N\pi z}{L}\right)\right]\,dz\right| \\
&\leq \frac{1}{2L}\lim_{N\to\infty}\left\{\left|\int_{-L}^{L}\left[\tilde{f}(x) - f(x+z)\right]\sin\left(\frac{N\pi z}{L}\right)\cot\left(\frac{\pi z}{2L}\right)\,dz\right|\right. \\
&\qquad\left. + \left|\int_{-L}^{L}\left[\tilde{f}(x) - f(x+z)\right]\cos\left(\frac{N\pi z}{L}\right)\,dz\right|\right\}.
\end{aligned} \tag{18}$$

Since $\tilde{f}(x) - f(x+z)$ is a piecewise C^1 function, then the second integral goes to zero in the limit. In the first integral, the function

$$\frac{\tilde{f}(x) - f(x+z)}{\tan\left(\frac{\pi z}{2L}\right)} \tag{19}$$

is piecewise C^1 everywhere except, maybe at $z = 0$. However, using L'Hopital's rule as $z \to 0$ we find that

$$\lim_{z\to 0}\frac{\tilde{f}(x) - f(x+z)}{\tan\left(\frac{\pi z}{2L}\right)} \to \frac{df(x)}{dx}\frac{2L}{\pi}, \tag{20}$$

which is a bounded function since $f(x)$ is C^1. Therefore the first integral also goes to zero in the limit.

2 Uniform Convergence of Fourier Series

The Fourier series of continuous, piecewise C^1 functions on the interval converge uniformly to the function. This means

$$\lim_{N\to\infty}\max_{-L\leq x\leq L}|f(x) - S_N(x)| \to 0. \tag{21}$$

Proof:

First we note that, we have already proved that the Fourier series of continuous functions $f(x)$ converge pointwise to the function $f(x)$, since the adjusted function $\tilde{f}(x)$ is equal to $f(x)$ for continuous functions.

Pointwise convergence means that $FS\ f(x) = f(x)$ so we can write

$$\begin{aligned}
\lim_{N\to\infty} \max_{-L\le x\le L} |f(x) - S_N(x)| &= \lim_{N\to\infty} \max_{-L\le x\le L} |FSf(x) - S_N(x)| \\
&= \lim_{N\to\infty} \max_{-L\le x\le L} \left| \sum_{n=N+1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right| \\
&\le \lim_{N\to\infty} \max_{-L\le x\le L} \sum_{n=N+1}^{\infty} |a_n| \left|\cos\left(\frac{n\pi x}{L}\right)\right| + |b_n| \left|\sin\left(\frac{n\pi x}{L}\right)\right| \\
&\le \lim_{N\to\infty} \sum_{n=N+1}^{\infty} |a_n| + |b_n|
\end{aligned} \tag{22}$$

If we can prove that this limit (the tail of series) vanishes then we have shown that the error goes to zero everywhere. The vanishing of this limit is the same as having

$$\sum_{n=1}^{\infty} |a_n| + |b_n| < \infty \tag{23}$$

i.e. the series converges if and only if the tail vanishes.

The coefficients of the Fourier series are

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \\
&= \frac{1}{L}\ f(x)\ \frac{L}{n\pi}\ \sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{1}{L}\ \frac{L}{n\pi}\int_{-L}^{L} f'(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx \\
&= -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx \\
|a_n| &\le \frac{1}{n\pi}A_n
\end{aligned} \tag{24}$$

where A_n are the Fourier cosine coefficients of the derivative function, $f(x)$. Since $f(x)$ is continuous and C^1, then $f'(x)$ is piecewise continuous. Similarly for b_n

$$|b_n| \le \frac{1}{n\pi}B_n \tag{25}$$

where B_n are the Fourier sine coefficients of the derivative function, $f(x)$.

Now, let $f'(x) = g(x)$ which is a piecewise continuous function. Therefore the coefficients A_n, B_n exist and the function $g(x)$ is L^2 integrable, meaning

$$\int_{-L}^{L} [g(x)]^2\ dx < \infty. \tag{26}$$

Bessel's inequality (which is very straightforward to derive) says that

$$\frac{A_0^2}{2} + \sum_{n=0}^{\infty} A_n^2 + B_n^2 \leq \int_{-L}^{L} [g(x)]^2 \; dx \tag{27}$$

and therefore

$$\sum_{n=0}^{\infty} A_n^2 + B_n^2 < \infty. \tag{28}$$

The Cauchy Schwarz Inequality:

For two sequences $\alpha = (\alpha_1, \alpha_2 .... \alpha_n)$ and $\beta = (\beta_1, \beta_2 .... \beta_n)$, define

$$\alpha \cdot \beta = \alpha_1 \beta_1 + ... + \alpha_n \beta_n \tag{29}$$

and

$$|\alpha| = \sqrt{\alpha \cdot \alpha}. \tag{30}$$

The Cauchy Schwarz inequality states that

$$|\alpha \cdot \beta| \leq \sqrt{|\alpha| \, |\beta|} \tag{31}$$

Proof:

First, note that $\alpha \cdot \alpha \geq 0 \; \forall \alpha$. Let r be any real number and notice that

$$\begin{aligned} 0 &\leq |\alpha + r\beta|^2 \\ &= (\alpha_1 + r\beta_1)^2 + ... + (\alpha_n + r\beta_n)^2 \\ &= \alpha_1^2 + ... + \alpha_n^2 + 2\,(\alpha_1\beta_1 + ... + \alpha_n\beta_n)\, r + \left(\beta_1^2 + ... + \beta_n^2\right)\, r^2 \\ 0 &\leq \alpha \cdot \alpha + 2\alpha \cdot \beta \, r + \beta \cdot \beta \, r^2 \end{aligned} \tag{32}$$

for all r. The right hand side of this expression describes a parabola in r which opens upward $(\beta \cdot \beta)$ and whose interior is always greater than zero. Therefore the parabola can only intersect the vertical axis at zero or one location. If it intersected at two locations then there would be an interval between the two roots $r_1 < r < r_2$ where the interior of the parabola would have values < 0 - which is not possible.

The condition that there be only one or no roots to a quadratic is that the discriminant of the quadratic formula be negative. The expression for the roots of this parabola is

$$r = -\alpha \cdot \beta \pm \sqrt{(\alpha \cdot \beta)^2 - |\alpha|^2 \, |\beta|^2} \tag{33}$$

and the discriminant must be zero or negative. Therefore

$$\begin{aligned} (\alpha \cdot \beta)^2 - |\alpha|^2 \, |\beta|^2 &\leq 0 \\ (\alpha \cdot \beta)^2 &\leq |\alpha|^2 \, |\beta|^2 \\ \alpha \cdot \beta &\leq |\alpha| \, |\beta| \end{aligned} \tag{34}$$

which is the Cauchy-Schwarz inequality.

Now let's substitute the value of a_n, b_n into the expression

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| + |b_n| &= \lim_{N\to\infty} \frac{1}{\pi} \sum_{n=1}^{N} \frac{|A_n|}{n} + \frac{|B_n|}{n} \\
&\leq \lim_{N\to\infty} \left\{ \frac{1}{\pi} \sqrt{\sum_{n=1}^{N} |A_n|^2} \sqrt{\sum_{n=1}^{N} \frac{1}{n^2}} + \frac{1}{\pi} \sqrt{\sum_{n=1}^{N} |B_n|^2} \sqrt{\sum_{n=1}^{N} \frac{1}{n^2}} \right\} \\
&\leq \frac{1}{\pi} \lim_{N\to\infty} \left\{ \sqrt{\sum_{n=1}^{N} |A_n|^2} + \sqrt{\sum_{n=1}^{N} |B_n|^2} \right\} \sqrt{\sum_{n=1}^{N} \frac{1}{n^2}} \\
&\leq \frac{1}{\pi} C \lim_{N\to\infty} \sqrt{\sum_{n=1}^{N} \frac{1}{n^2}}.
\end{aligned} \tag{35}$$

The logic of the last line is that the sequences A_n and B_n are each square integrable - which means that the sum of the square roots can be pulled in front of the limit. It remains to prove that the limit of the remaining sum exists,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} < \infty \tag{36}$$

which we will do by actually evaluating the sum. Consider the function

$$f(x) = x; \tag{37}$$

its Fourier Sine series has coefficients

$$\begin{aligned}
b_n &= \frac{1}{L} \int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{1}{L} \left(-\frac{L}{n\pi}\right) \left[L \cos\left(\frac{n\pi L}{L}\right) - (-L) \cos\left(-\frac{n\pi L}{L}\right) - \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx \right] \\
&= -\left(\frac{2L}{n\pi}\right) \cos(n\pi) \\
&= \frac{2L}{\pi} \frac{(-1)^{n+1}}{n}
\end{aligned} \tag{38}$$

and is written

$$FS\ f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{.} L\right) \tag{39}$$

Now Bessel's inequality applied to $f(x)$ implies

$$\begin{aligned} \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} &\leq \frac{1}{L} \int_{-L}^{L} x^2 \, dx \\ \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} &\leq \frac{2L^3}{3L} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &\leq \frac{\pi^2}{6}. \end{aligned} \tag{40}$$

So we conclude that

$$\sum_{n=1}^{\infty} |a_n| + |b_n| \leq C \, \frac{\pi}{6}. \tag{41}$$

Since this series coverges then its tail must vanish, i.e

$$\lim_{N \to \infty} \sum_{n=N+1}^{\infty} |a_n| + |b_n| \to 0. \tag{42}$$

This completes the proof of uniform convergence.