Derivation and *Explanation* of Rayleigh and Fjortoft's necessary conditions for shear flow instability.

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1 Rayleigh's equation for shear flows

I learned about these stability criteria when in graduate school ever so long ago. It is with some chagrin that I confess that I never fully understood the implications of Fjortoft's criterion. Herein I undertake to derive these criteria - in the standard way - but more importantly, I hope to use examples of flows which explain (at least in my understanding) these criteria.

We begin with the 2-D Euler equations describing linearized perturbations to a shear flow, $\bar{U}(y)$,

$$\begin{aligned} u_t + \bar{U}u_x + v\bar{U}_y + p_x &= 0 \\ v_t + \bar{U}v_x + p_y &= 0 \\ u_x + v_y &= 0. \end{aligned} \tag{1}$$

Taking the curl of the momentum equations and using the vorticity $\omega = v_x - u_y$ we find

$$\omega_t + \bar{U}\omega_x - u_x\bar{U}_y - v_y\bar{U}_y - v\bar{U}_{yy} = 0$$

which, upon using incompressibility and the stream function $u = -\psi_y, v = \psi_x$ and $\omega = \nabla^2\psi$ yields

$$\left(\partial_t + \bar{U}\partial_x\right)\nabla^2\psi - \bar{U}_{yy}\psi_x = 0. \tag{2}$$

As is standard, we assume normal modes, $\psi = \Psi(y)e^{ik(x-ct)} + c.c.$, so that imaginary c corresponds to instability. Rayleigh's equation is

$$\left(\bar{U} - c\right)\left(D^2 - k^2\right)\Psi - \bar{U}_{yy}\Psi = 0. \tag{3}$$

If the background flow is not twice differentiable, we should rewrite (3) in a form which allows us to describe jump conditions,

$$\left[\left(\bar{U} - c\right)\Psi_y - \bar{U}_y\Psi\right]_y - k^2\left(\bar{U} - c\right)\Psi = 0. \tag{4}$$

This form of Rayleigh's equation actually contains a lot of mathematical content. The background vorticity $-\bar{U}_y$ must exist everywhere, but it need not be differentiable. At points,

$y = y_0$, where it is not differentiable we seek solutions where Ψ is continuous. Furthermore, integrating this equation across $y = y_0$ yields the jump condition

$$\left[\left(\bar{U} - c\right)\Psi_y - \bar{U}_y \Psi\right]\Big|_{y_0-\epsilon}^{y_0+\epsilon} = 0. \tag{5}$$

So, both Ψ and (5) must be continuous across a vorticity jump.

2 Integrals of Rayleigh's equation

Integral relations are useful for determining necessary conditions for instability. There is no sufficient condition for instability which is known. To get a sufficient condition, you have to explicitly solve the eigenvalue problem for a particular choice of $\bar{U}(y)$. Practically speaking, however, if you find enough necessary conditions for instability, the union of these conditions provides *essentially* sufficient condition for insability.

Shear flows have both continuous and discrete spectrum. The continuous spectrum is always stable - but it has singular eigenvalues (i.e. they may not be differentiable). If we define the range of $\bar{U}$ as

$$\mathcal{R}[\bar{U}] = [U_{\min}, U_{\max}] \tag{6}$$

which are all the values that the background velocity takes on. The continuous spectrum consists of all wave speeds, c in this range since it arises due to the vanishing of $\bar{U} - c$ in the range of the flow. I will show some eigenfunctions later in this writeup.

The discrete spectrum contains both stable and unstable waves. For now, I am only interested in the discrete spectrum in order to understand instability. Considering complex valued Ψ and $c = c_r + ic_i$ we multiple (3) by $\Psi^*/\left(\bar{U} - c\right)$ and integrate over the domain $\mathcal{D} \in \mathbf{R}^2$ to find

$$\int_{\mathcal{D}} \Psi^* \left(D^2 - k^2\right) \Psi \, dy = \int_{\mathcal{D}} \frac{\bar{U}_{yy}|\Psi|^2}{\bar{U} - c} \, dy. \tag{7}$$

Integrating the first term on the left hand side by parts, we will assume that either the boundary conditions are compatible with $\Psi = 0$ and $D\Psi = 0$ or that the eigenfunctions are compactly supported- which means that they vanish far away from the shear. Multiplying the top and bottom of the denominator on the right hand side by $\bar{U} - c^*$ we find

$$-\int_{\mathcal{D}} |D\Psi|^2 + k^2|\Psi|^2 \, dy = \int_{\mathcal{D}} \frac{\bar{U}_{yy}\left(\bar{U} - c^*\right)|\Psi|^2}{|\bar{U} - c|^2} \, dy. \tag{8}$$

Upon expanding out the right hand side numerator we write

$$-\int_{\mathcal{D}} |D\Psi|^2 + k^2|\Psi|^2 \, dy = \int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}\left(\bar{U} - c_r\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2}\right] |\Psi|^2 \, dy \; + \; i\int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}\, c_i}{\left(\bar{U} - c_r\right)^2 + c_i^2}\right] |\Psi|^2 \, dy \tag{9}$$

This integral has the property that the left hand side is real and negative for any non-trivial Ψ while the right hand side explicitly manifests its real and imaginary parts; the first term is purely real, the second is purely imaginary.

2.1 Rayleigh's Criterion

Since the imaginary part of the right hand side of (9) must vanish, this implies that the second integral on the right hand side is zero.

Either $c_i = 0$, so the second term vanishes, and the flow is stable. On the other hand, if $c_i \neq 0$, the flow is unstable. If $c_i \neq 0$, then the denominator in the integral also never vanishes. Therefore, if a flow is unstable, its eigenfunction must have the property that

$$\int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy = 0. \tag{10}$$

The denominator of the integrand is clearly positive, as is $|\Psi|^2$. So the only term in this integral which has the possibility of changing sign is $\bar{U}_{yy}$.

Therefore, a necessary condition for instability is that $\bar{U}_{yy}$ changes sign within the flow and so it must be zero somewhere within the flow; this is Rayleigh's criterion, i.e. that the flow must have an inflection point.

Another way to think of this is in terms of the vorticity of the background flow,

$$\bar{\Omega} = -\bar{U}_y. \tag{11}$$

The statement of Rayleigh's criterion is equivalent to the statement that $\bar{\Omega}_y$ must change sign within the flow and therefore

$$\bar{\Omega}_y = 0 \tag{12}$$

in some finite interval in the flow. Therefore, a necessary condition for instability is that the background flow have a vorticity extremum within the domain.

As a matter of notation, we define y_I as the location of the inflection point, $\bar{U}_{yy}(y_I) = 0$ and $U_I = \bar{U}(y_I)$ as the speed of the flow at the inflection point.

2.2 Fjortoft's Criterion

The real part of the integral relation (9) is

$$-\int_{\mathcal{D}} |D\Psi|^2 + k^2|\Psi|^2 \, dy = \int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}\left(\bar{U} - c_r\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy. \tag{13}$$

Not knowing anything about the eigenfunctions or eigenvalues, we can only conclude that for an unstable mode, the eigenfunction and eigenvalue satisfies

$$\int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}\left(\bar{U} - c_r\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy < 0. \tag{14}$$

In particular, we note that for an unstable flow, the second term in the numerator (the one multiplying c_r) is already zero by virtue of the integral (10) - i.e. it must be zero due to Rayleigh's criterion. Therefore, we can replace c_r in the numerator by *any constant*, U_* and still have the necessary condition that the integral be less than zero, i.e.

$$\int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}\left(\bar{U} - U_*\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy < 0 \quad \forall\, U_* \in \mathbf{R}. \tag{15}$$

The correct choice of U_* will give us a more restrictive condition than Rayleigh's criterion - this will be Fjortoft's criterion. How do we find the correct choice of U_*? I used to have trouble understanding this part of the argument and I hope I can convey a better understanding of it hre.

Imagine a flow that satisfies Rayleigh's criterion, so that an inflection point exists and we wish to consider if (15) provides any further *simple* necessary conditions for instability. [Notice that the integral itself is a necessary condition for instability, but it involves the unknown eigenvalues and eigenfunction.] Furthermore, we would like to choose the value of U_* that provides the most restrictive necessary condition for instability.

The first point is that we have no a priori information about $|\Psi|^2$ meaning that it can be as localized around the inflection point as we would like. So too, we have little information about the denominator - other than it is positive and non-vanishing. This leads us to look at the behavior of the numerator around the inflection point.

The argument is now simple (in my mind), if one considers a **small neighborhood around** the inflection point. For an arbitrary choice of U_* (we are allowed to choose any U_*), $\bar{U} - U_*$ is sign definite (either positive or negative) in such a small neighborhood. Since $\bar{U}_{yy}$ changes sign in that small neighborhood, then the integrand is not sign definite. So, for most choices of U_*, the integrand changes sign near $y = y_I$ and (15) does not provide a more restrictive condition than Rayleigh's criterion.

However, if we choose

$$U_* = U_I = \bar{U}(y_I) \tag{16}$$

where $y = y_I$ is the location of the inflection point, then both factors in the numerator are (approximately) linear functions of $y - y_I$ near the inflection point. Therefore their product is proportional to

$$(y - y_I)^2 \tag{17}$$

near the inflection point. The Fjortoft necessary condition for instability is that the porportionality constant must be negative!

Let's argue this in another way. Start by considering flows with one inflection point. It seems reasonable that we can restrict attention to these situations by simply restricting the domain of integration to such regions, since we are only establishing necessary conditions for instability.

If we choose U_* outside the range of $\bar{U}$ then $(\bar{U} - U_*)$ is sign definite, just like the denominator and just like $|\Psi|^2$. In this case, we conclude that the only condition is that $\bar{U}_{yy}$ must change sign, which is still Rayleigh's criterion. If we choose U_* within the range of $\bar{U}$ then there exists at least one y_* where $\bar{U} - U_*$ changes sign, i.e. $U(y_*) = U_*$.

To fix ideas, let's choose $\bar{U}_{yy}$ to be a function with one zero at $y = y_I$ on the domain $a < y < b$. This means $-\bar{U}_y$ (the vorticity) contains only one extremum at $y = y_I$. It *does not* mean the velocity, $\bar{U}$ is monotonic - the velocity can change sign.

Start by assuming $y_I < y_*$ and, using a, b to denote the endpoints of the domain (or at least the endpoints of the region where $\bar{U}$ is monotonic and the vorticity has one we can split up the integral as follows

$$\int_a^{y_I} + \int_{y_I}^{y_*} + \int_{y_*}^{b} < 0. \tag{18}$$

Since one or other of the terms in the numerator changes sign at each of these internal splitting points, we conclude that the first and third integrals have the same sign, whereas the second integral has the opposite sign. The same situation would occur if $y_* < y_I$. In both of these cases, the integrals afford a lot of self-cancellation, and are not successful in creating a more restrictive necessary condition than Rayleigh's crieterion.

However, if $y_* = y_I$ then $U_* = U_I$ and the middle integral vanishes (because one is integrating over no range). Again, the first and third integrals have the same sign, so any condition they create is more restrictive than any case when $y_* \neq y_I$.

Therefore, the integral in (15) can be written

$$\int_{\mathcal{D}} \left[\frac{\bar{U}_{yy}\left(\bar{U} - U_I\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy < 0 \tag{19}$$

which is the more restrictive necessary condition for instability. Again, not knowing the eigenvalues or eigenfunctions, we are left to conclude that the stricter necessary condition for instability is

$$\bar{U}_{yy}\left(\bar{U} - U_I\right) < 0 \tag{20}$$

for a range of y values in the region where the flow is monotonic and the vorticity has only one extremum (i.e. the flow has only one inflection point). However, we also notice that these restrictions (domain where flow is monotonic and has a single vorticity extremum) imply that the condition in (20) is satisfied throughout this part of the domain except at $y = y_I$.

Writing (20) in terms of the vorticity we find that a necessary condition for instability is

$$\bar{\Omega}_y\left(\bar{U} - U_I\right) > 0 \tag{21}$$

for subregions of the flow where it is monotonic and has only one extremum of vorticity (i.e. $\bar{\Omega}_y = 0$ at only one point in the region). The condition (21) says that where the vorticity is increasing, the flow must be faster than the flow at the vorticity maximum (i.e. the inflection point of the velocity field), and where the vorticity is decreasing, the flow must be slower than the flow at the vorticity maximum (the inflection point of the velocity field).

Another way of thinking about this criterion (21) is using Taylor expansions. In a neighborhood of the inflection point (i.e. in a neighborhood of the vorticity extremum) we can write

$$\begin{aligned} U - U_I &= -\bar{\Omega}(y_I)\,(y - y_I) - \bar{\Omega}_y(y_I)\frac{(y - y_I)^2}{2} - \bar{\Omega}_{yy}(y_I)\frac{(y - y_I)^3}{6} + ... \\ \text{and} \qquad \bar{\Omega}_y &= \bar{\Omega}_y(y_I) + \bar{\Omega}_{yy}(y_I)\,(y - y_I) + ... \end{aligned} \tag{22}$$

However, since y_I is a vorticity maximum, then $\bar{\Omega}_y(y_I) = 0$. Therefore the product in (21) becomes

$$-\bar{\Omega}(y_I)\bar{\Omega}_{yy}(y_I)\,(y - y_I)^2 + \mathcal{O}((y - y_I)^3) > 0. \tag{23}$$

So, close enough to $y = y_I$ the first term will dominate. The necessary condition for instability is therefore

$$\bar{\Omega}(y_I)\bar{\Omega}_{yy}(y_I) < 0. \tag{24}$$

This is still not a very intuitive criterion. Let's try to massage it a little by using the following fact, which simply amounts to two applications of the product rule

$$\frac{d^2}{dy^2}[AB] = \frac{d^2 A}{dy^2} B + 2\frac{dA}{dy}\frac{dB}{dy} + A\frac{d^2 B}{dy^2} \tag{25}$$

and let $A = B = \bar{\Omega}$, so that

$$\left[\frac{\bar{\Omega}^2}{2}\right]_{yy} = \bar{\Omega}\,\bar{\Omega}_{yy} + \left[\bar{\Omega}_y\right]^2 \tag{26}$$

Evaluating this expression at $y = y_I$ and recognizing that $\bar{\Omega}_y(y_I) = 0$, then equation (24) simply becomes

$$\left[\bar{\Omega}^2\right]_{yy} < 0, \qquad \text{at} \quad y = y_I. \tag{27}$$

I very much like this formulation of Fjortoft's criterion because it makes no reference to the sign of the vorticity, only its magnitude. It clearly states that at the inflection point in the velocity field, which is an extremum of the vorticity the amplitude (squared) of the vorticity must achieve a *local maximum* in order for the flow to be unstable. In my opinion, this is the physical content of Rayleigh and Fjortoft's criteria; the vorticity must have an extremum, and that extremum must be such that the magnitude of the vorticity is maximum, in order for their to be instability.

2.2.1 Example of Fjortoft's criterion

Let's create a model flow in which $\bar{U}$ is monotonically increasing with y, which means that $\bar{\Omega}$ is everywhere negative. Consider the vorticity profile

$$\bar{\Omega} = -\left\{\Omega + \alpha\,\Theta(L - |y|)\left[1 - \frac{|y|}{L}\right]\right\} \tag{28}$$

where Ω and α are parameters. The function $\Theta(s)$ is the Heaviside function which takes value 1 at positive argument, 0 at negative argument. This flow has a piecewise linear vorticity, with an extremum at $y = y_I = 0$ (see figure 1).

A monotonically increasing velocity profile requires $\bar{\Omega} < 0$ for all y, which implies for our parameters, $\Omega > 0$ and $\Omega + \alpha \geq 0$. We don't yet know the sign of α for which the flow will be unstable.

Integrating this vorticity allows us to calculate $\bar{U}$

$$\begin{aligned}
\bar{U} - U_I &= -\int_0^y \bar{\Omega}(y')\, dy' \\
&= \int_0^y \left\{ \Omega + \alpha\Theta(L^2 - y'^2)\left[1 - \frac{|y'|}{L}\right]\right\} dy' \\
&= \Omega y + \alpha \begin{cases} \frac{L}{2} & y \geq L \\ y - \frac{y^2}{2L} & 0 \leq y < L \\ y + \frac{y^2}{2L} & -L < y < 0 \\ -\frac{L}{2} & y \leq -L \end{cases} \\
&= \Omega y + \alpha \begin{cases} \frac{\sigma(y) L}{2} & |y| \geq L \\ y - \frac{y|y|}{2L} & |y| < L \end{cases}
\end{aligned} \tag{29}$$

where $\sigma(y) = \pm 1$ is the sign function. The derivative of the vorticity is

$$\bar{\Omega}_y = \begin{cases} \frac{\sigma(y)\,\alpha}{L} & |y| \leq L \\ 0 & |y| > L \end{cases} \tag{30}$$

Fjortoft's necessary condition for instability (21) is thus

$$\alpha \left[(\Omega + \alpha) \frac{|y|}{L} - \frac{\alpha y^2}{2L^2} \right] > 0 \quad \text{on} \quad |y| \leq L. \tag{31}$$

Since it is a symmetric function, let's focus on $y > 0$ and let $y = L\eta$ so that

$$\alpha\eta \left[(\Omega + \alpha) - \frac{\alpha\eta}{2} \right] > 0 \quad \text{on} \quad 0 < \eta \leq 1. \tag{32}$$

The quantity in parenthesis varies from

$$\Omega + \frac{\alpha}{2} = \frac{\Omega}{2} + \frac{\Omega + \alpha}{2} > 0 \quad \text{to} \quad \Omega + \alpha \geq 0 \tag{33}$$

and is, thus, always positive. Therefore, Fjortoft's necessary condition for instability requires

$$\alpha > 0. \tag{34}$$

In figure 1 are plotted two vorticity profiles and their corresponding flows. Both flows satisfy Rayleigh's necessary condition for instability. The flow on the left satisfies Fjortoft's condition for instability while that on the right does not.

Another way to interpret this result is that, in an unstable flow, the shear $\bar{U}_y$ at the inflection point is greater than the average shear throughout the domain. In this case, the average shear over the (infinite) domain is Ω whereas the shear at the inflection point is $\Omega + \alpha$. For the latter to be greater than the former requires $\alpha > 0$.

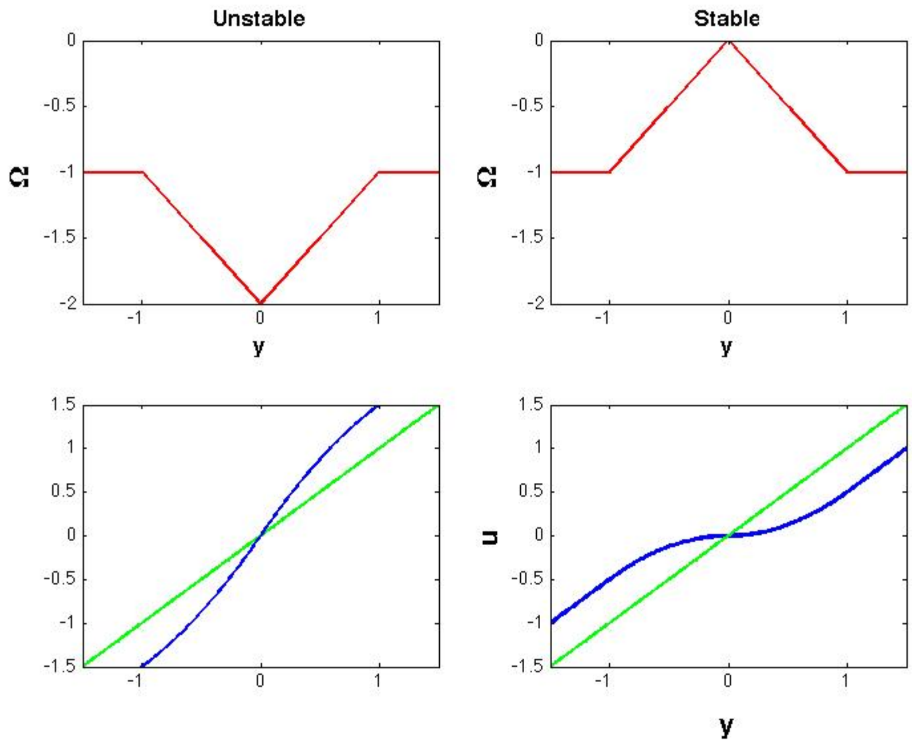

Figure 1: The flow on the left satisfies Fjortoft's condition for instability while that on the right does not. The asymptotic value of the shear flow is shown in green (asymptotically parallel).

3 (Louie) Howard's Semicircle Theorem

I had the great honor of knowing and learning some from Lou Howard - and his contribution to fluid dynamics and physical oceanography was massive. His semi-circle theorem for the unstable eigenvalues of Rayleigh's equation has been used in many more complicated contexts since he derived it - and was an inspiration for some models of shear flow turbulence.

Returning to Rayleigh's equation (3) we change variables to

$$\Phi = \frac{\Psi}{\bar{U} - c} \tag{35}$$

so that

$$\begin{aligned}
\left(\bar{U} - c\right)\left(D^2 - k^2\right)\left[\left(\bar{U} - c\right)\Phi\right] - \bar{U}_{yy}\left(\bar{U} - c\right)\Phi = 0 \\
\left(\bar{U} - c\right)D^2\left[\left(\bar{U} - c\right)\Phi\right] - \bar{U}_{yy}\left(\bar{U} - c\right)\Phi - k^2\left(\bar{U} - c\right)^2\Phi = 0 \\
\left(\bar{U} - c\right)\left[\left(\bar{U} - c\right)D^2\Phi + 2D\left(\bar{U} - c\right)D\Phi\right] - k^2\left(\bar{U} - c\right)^2\Phi = 0
\end{aligned}$$

which yields the self-adjoint form of Rayleigh's equation

$$D\left[\left(\bar{U} - c\right)^2 D\Phi\right] - k^2\left(\bar{U} - c\right)^2\Phi = 0. \tag{36}$$

Multiplying by Φ^* and integrating by parts yields

$$\int_{\mathcal{D}} \left(\bar{U} - c\right)^2 \left[|D\Phi|^2 + k^2|\Phi|^2\right] \; dy = 0. \tag{37}$$

Defining $d\Gamma = [|D\Phi|^2 + k^2|\Phi|^2] \; dy$ and separating the real and imaginary parts again we find

$$\int_{\mathcal{D}} \left[\left(\bar{U} - c_r\right)^2 - c_i^2\right] \; d\Gamma + 2ic_i \int_{\mathcal{D}} \left(\bar{U} - c_r\right) \; d\Gamma = 0. \tag{38}$$

Both the real and imaginary parts of (38) must vanish so, if $c_i \neq 0$ (instability) then

$$\int_{\mathcal{D}} \left(\bar{U} - c_r\right) \; d\Gamma = 0. \tag{39}$$

The expression in parenthesis (the Doppler shifted speed) must change sign within the domain, therefore

$$\bar{U}_{\min} < c_r < \bar{U}_{\max} \tag{40}$$

for bounded flows in a bounded domain. The real part of (38) is

$$\int_{\mathcal{D}} \left[\left(\bar{U} - c_r\right)^2 - c_i^2\right] \; d\Gamma = 0. \tag{41}$$

The following inequality is straightforward

$$\int_{\mathcal{D}} \left(\bar{U} - \bar{U}_{\min}\right)\left(\bar{U}_{\max} - \bar{U}\right) \; d\Gamma \geq 0 \tag{42}$$

since both terms in parentheses are positive, by construction. Adding (41) to (42) yields

$$\begin{aligned} \int_{\mathcal{D}} \left[\bar{U}^2 - 2c_r\bar{U} + c_r^2 - c_i^2 - \bar{U}^2 + \bar{U}\left(\bar{U}_{\max} + \bar{U}_{\min}\right) - \bar{U}_{\min}\bar{U}_{\max}\right] \; d\Gamma > 0 \\ \int_{\mathcal{D}} \left[-2c_r^2 + c_r^2 - c_i^2 + c_r\left(\bar{U}_{\max} + \bar{U}_{\min}\right) - \bar{U}_{\min}\bar{U}_{\max}\right] \; d\Gamma > 0 \end{aligned} \tag{43}$$

where I used (39) to remove the linear terms in $\bar{U}$. Further simplifying we find

$$\int_{\mathcal{D}} \left[-c_r^2 - c_i^2 + c_r\left(\bar{U}_{\max} + \bar{U}_{\min}\right) - \bar{U}_{\min}\bar{U}_{\max}\right] \; d\Gamma > 0 \tag{44}$$

and we can drop the integral with respect to Γ since it is clearly positive,

$$\begin{aligned} -c_r^2 - c_i^2 + c_r\left(\bar{U}_{\max} + \bar{U}_{\min}\right) - \bar{U}_{\min}\bar{U}_{\max} > 0 \\ c_r^2 - c_r\left(\bar{U}_{\max} + \bar{U}_{\min}\right) + \bar{U}_{\min}\bar{U}_{\max} + c_i^2 < 0 \\ c_r^2 - c_r\left(\bar{U}_{\max} + \bar{U}_{\min}\right) + \frac{\left(\bar{U}_{\max} + \bar{U}_{\min}\right)^2}{4} - \frac{\left(\bar{U}_{\max} + \bar{U}_{\min}\right)^2}{4} + \bar{U}_{\min}\bar{U}_{\max} + c_i^2 < 0 \\ \left[c_r - \frac{\left(\bar{U}_{\max} + \bar{U}_{\min}\right)}{2}\right]^2 + c_i^2 < \left(\frac{\bar{U}_{\max} - \bar{U}_{\min}}{2}\right)^2. \end{aligned} \tag{45}$$

Therefore, the eigenvalue lies in a semicircle in the complex plane whose real part is centered at the average shear speed and whose radius is half the difference of the maximum and minimum velocities.

This creates a maximum bound on the imaginary part of the speed - i.e. the growth rate divided by the wavenumber. Unfortunately, it does not tell us how the growth rates vary as a function of the wavenumber, k.

4 Shear Flow on an equatorial β-plane

Barotropic instability on a β-plane is governed by the equations

$$\begin{aligned} u_t + \bar{U} u_x + v \bar{U}_y - \beta y v + p_x &= 0 \\ v_t + \bar{U} v_x + \beta y u + p_y &= 0 \\ u_x + v_y &= 0. \end{aligned} \tag{46}$$

This result also works on a mid-latitude β-plane because the constant planetary vorticity drops out of the problem upon taking derivatives. Taking the curl of the momentum equations yields the vorticity equation

$$\omega_t + \bar{U}\omega_x - \left[\bar{U}_{yy} - \beta\right] v = 0, \tag{47}$$

which, upon using incompressibility and the stream function yields

$$\left(\partial_t + \bar{U}\partial_x\right) \nabla^2 \psi - \left[\bar{U}_{yy} - \beta\right] \psi_x = 0, \tag{48}$$

and, as before, the corresponding eigenvalue problem

$$\left(\bar{U} - c\right) \left(D^2 - k^2\right) \Psi - \left[\bar{U}_{yy} - \beta\right] \Psi = 0. \tag{49}$$

We can reproduce the steps to derive the integral in (9), but the result is simply

$$-\int_{\mathcal{D}} |D\Psi|^2 + k^2 |\Psi|^2 \, dy = \int_{\mathcal{D}} \left[\frac{\left(\bar{U}_{yy} - \beta\right)\left(\bar{U} - c_r\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy + i \int_{\mathcal{D}} \left[\frac{\left(\bar{U}_{yy} - \beta\right) \; c_i}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy \tag{50}$$

4.1 Rayleigh's Criterion on the β-plane

As before, the imaginary part of the integral must vanish and $c_i \neq 0$ for instability. This means that $\bar{U}_{yy} - \beta$ changes sign somewhere in the flow and therefore

$$\bar{U}_{yy} - \beta = 0 \tag{51}$$

at some point in the flow, $y = y_\beta$ - not the inflection point of the flow, but rather the location of the zero of (58) . Defining the total vorticity as

$$\bar{\Lambda} = -\bar{U}_y + \beta y = -\left[\bar{U} - \frac{\beta y^2}{2}\right]_y \tag{52}$$

then Rayleigh's condition for instability is that

$$\bar{\Lambda}_y = 0 \tag{53}$$

somewhere in the flow; i.e. the total vorticity, planetary plus flow, must have an extremum.

4.2 Fjortoft's Criterion on the β-plane

Using the real part of (50) we conclude that an additional, necessary condition for instability is

$$\int_{\mathcal{D}} \left[\frac{\left(\bar{U}_{yy} - \beta\right)\left(\bar{U} - U_*\right)}{\left(\bar{U} - c_r\right)^2 + c_i^2} \right] |\Psi|^2 \, dy < 0 \tag{54}$$

for any U_*. The same arguments apply as in the previous example, which is to say that the most restrictive example occurs when $U_* = \bar{U}(y_\beta) \equiv \bar{U}_\beta$. Furthermore, the numerator must be negative somewhere within the domain, and if we restrict the domain to regions of monotonic velocity with only one extremum of total vorticity, then the necessary condition for instability (over this region) is

$$\left(\bar{U}_{yy} - \beta\right)\left(\bar{U} - U_\beta\right) < 0, \tag{55}$$

for some y near y_β, which is the generalization of Fjortoft's criterion.

4.2.1 An example flow

Like before we want to study the velocity field locally around the point y_β

$$\begin{aligned} U - U_\beta &= -\bar{\Omega}(y_\beta)\,(y - y_\beta) - \bar{\Omega}_y(y_\beta)\frac{\left(y - y_\beta\right)^2}{2} - \bar{\Omega}_{yy}(y_\beta)\frac{\left(y - y_\beta\right)^3}{6} + ... \\ \text{and} \qquad \bar{\Omega}_y &= \bar{\Omega}_y(y_\beta) + \bar{\Omega}_{yy}(y_\beta)\,(y - y_\beta) + ... \end{aligned} \tag{56}$$

Now the (negative of the) total vorticity is

$$\begin{aligned} \bar{U}_{yy} - \beta &= -\bar{\Omega}_y - \beta \\ &= -\beta - \bar{\Omega}_y(y_\beta) - \bar{\Omega}_{yy}(y_\beta)\,(y - y_\beta) + ... \end{aligned} \tag{57}$$

Now, Rayleigh's criterion requires the vanishing of this total vorticity at some $y = y_\beta$ (by definition of y_β). Therefore

$$\bar{\Omega}_y(y_\beta) = -\beta \tag{58}$$

expresses Rayleigh's criterion and locates the latitude at which there is the potential for instability. Let's imagine that this criterion (58) is satisfied, what does Fjortoft's criterion imply? We compute

$$\begin{aligned} \left(\bar{U}_{yy} - \beta\right)\left(\bar{U} - U_\beta\right) &= \left[-\bar{\Omega}_{yy}(y_\beta)\,(y - y_\beta) + ...\right]\left[-\bar{\Omega}(y_\beta)\,(y - y_\beta) + ...\right] \\ &= \bar{\Omega}(y_\beta)\bar{\Omega}_{yy}(y_\beta)\,\left(y - y_\beta\right)^2 + ... \end{aligned} \tag{59}$$

As before, this quantity must be negative for some interval around y_β, therefore Fjortoft's criterion implies

$$\bar{\Omega}(y_\beta)\bar{\Omega}_{yy}(y_\beta) < 0. \tag{60}$$

Again this is not the most intuitive of results. If instead we follow the same steps that led to equation (27) we use

$$\bar{\Omega}\,\bar{\Omega}_{yy} = \left[\frac{\bar{\Omega}^2}{2}\right]_{yy} - \left[\bar{\Omega}_y\right]^2. \tag{61}$$

Evaluating this expression at y_β, using (58) and (60) we find Fjortoft's necessary condition for instability on a β plane yields

$$\left[\bar{\Omega}^2\right]_{yy} < 2\beta^2. \tag{62}$$

We see that in the limit $\beta \to 0$ we recover the previous result. On the other hand, this result on a β plane has a different consequence. Assuming $\beta > 0$ (as we can do without loss of generality by choosing the direction of y) then the vorticity gradient must be sufficiently negative, according to Rayleigh's criterion, at some y_β. This is not a vorticity extremum - and Fjortoft's criterion no longer requires that it be a maximum of the vorticity magnitude. Instead, Fjortoft implies that the square magnitude not have significantly large positive curvature at y_β.

4.2.2 A specific example

Again consider a monotonic velocity field, but a vorticity field with an extremum

$$\bar{\Omega} = -\left\{\Omega + \alpha \,\text{sech}^2\left(\frac{y}{L}\right)\right\} \tag{63}$$

I choose this form with $\Omega > 0$ so that the velocity, $\bar{U}$ is monotonically increasing away from the support of the hyperbolic secant. The velocity is

$$\bar{U} = \Omega y + \alpha L \,\tanh\left(\frac{y}{L}\right) \tag{64}$$

which is a nice simple function. Now Rayleigh's necessary condition implies

$$\frac{2\alpha}{L}\,\text{sech}^2\left(\frac{y_\beta}{L}\right)\tanh\left(\frac{y_\beta}{L}\right) = -\beta \tag{65}$$

So for $\alpha > 0$ there is a possible solution with $y_\beta < 0$, and vice versa for negative α. Now the functions on the left hand side are bounded by one and negative one, so that there is a necessary condition for instability. Lets define

$$\gamma \equiv \frac{\beta L}{2\alpha} \quad \text{and} \quad T = \tanh\left(\frac{y_\beta}{L}\right). \tag{66}$$

Rayleigh's criterion becomes

$$\left(1 - T^2\right) T = -\gamma \tag{67}$$

for a given value of $0 < \gamma < 1$ and $-1 < T < 1$. Notice that the left hand side is a cubic in T whose maximum (minimum) occurs at $T = \pm\frac{1}{2}$ and has value $\frac{3}{8}$. Therefore, for an instability to exist,

$$\begin{aligned} 0 < \gamma < \frac{3}{8} \\ \Longrightarrow \alpha > \frac{4}{3}\beta L. \end{aligned} \tag{68}$$

If this is satisfied, the equation (67) gives the value of y at which instability can occur.

The application of Fjortoft's criterion here is more complicated, which I will defer to another day.