The Eady model of baroclinic instability using Surface Quasi-Geostrophy

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March 3, 2019

1 Quasi-Geostrophy and its boundary conditions

We begin by focusing on quasi-geostrophic dynamics on an f-plane with constant buoyancy frequency and no forcing. The potential vorticity is given from the stream function by

$$q = \psi_{xx} + \psi_{yy} + \left[\frac{f^2}{N^2}\psi_z\right]_z . \tag{1}$$

The potential vorticity is advected by the horizontal velocity field

$$q_t + uq_x + vq_y = 0 \tag{2}$$

and the velocity field is determined from the stream function by

$$u = -\psi_y, \quad v = \psi_x. \tag{3}$$

Although we do not need the pressure and potential temperature for the mathematics of the problem, it is useful to write them in order to understand the physics,

$$p = f\psi, \quad \theta = f\psi_z. \tag{4}$$

In order to compute a stream function, not only must q be advected, but then the Laplacian must be inverted for ψ. There are two natural boundary conditions, he first being that the pressure is fixed on a (horizontal) boundary. This is natural for a free surface and amounts to a Dirichlet condition on ψ. We can imagine that this is an appropriate condition for infinite domains.

A second natural condition is that the temperature is fixed on a boundary - whether this is physical or not is depends on the setup, but I could see that it might be appropriate to baroclinic instability over an ocean.

A third boundary condition is the one used in the Eady problem. The traditional argument goes as follows. At a rigid horizontal boundary, the no-penetration boundary condition implies that the vertical velocity is zero. Using the buoyancy (or potential temperature) equation in the original primitive equations from which QG was derived, we can set $w = 0$.

Therefore, the potential is advected in the horizontal direction, by the geostrophically balanced flow along such a boundary

$$\theta_t + u\theta_x + v\theta_y = 0 \quad \text{along a rigid boundary.} \tag{5}$$

And the closure is given by the definition of u, v, θ in terms of ψ.

In my opinion, there are 2 glaring problems with this boundary condition. The first problem is associated with the fact that the vertical velocity in QG is already zero at lowest order. Enforcing $w = 0$ in this manner is actually a way of enforcing the first order vertical velocity - which does not participate in the QG dynamics - to be zero. In order to do this, we have to use an equation which is not part of the QG theory - although it was part of the original equations. In fact, the horizontal advection of temperature equation actually arises as forcing to the first order vertical velocity in the asymptotic derivation of QG and setting it to zero makes this higher order vertical velocity be zero. Estimating the size of this velocity term, we see that it is about 0.5 cm/s in the troposphere. This small amplitude causes me to question the utility of such a strict constraint.

The second problem I have with this boundary condition is more mathematical in origin. We solve an equation like QG by specifying q and some boundary conditions for ψ. We then compute ψ by inverting the Laplacian, compute the velocity through derivatives of ψ, and then transport q for a small time, Δt and repeat.

We cannot carry out this procedure if we use the boundary condition in (5) since the temperature is transported, but not specified, on the boundary. So using (5) we cannot specify an initial condition for q, but must specify an initial condition for $\psi = p/f$. From this we can compute both θ and q through (4) and (1).

While neither of this issues mean that the boundary condition is incorrect, they do help us understand what the issue with the boundary condition can be and how to understand the different kinds of baroclinic instability that can arise. The Eady problem (as we will see below) describes an instability that has no potential vorticity inside the domain - it rests on the buoyancy dynamics of the interfaces alone.

2 Surface Quasi-Geostrophy

I will show that SQG can be thought of as an advection equation for a temperature jump across an interface. Let's consider the QG equation with a δ-distribution potential vorticity

$$q = -2\Sigma(x, y)\,\delta(z). \tag{6}$$

Solving for the stream function we find

$$\psi_{xx} + \psi_{yy} + \left[\frac{f^2}{N^2}\psi_z\right]_z = 0 \quad z \neq 0 \tag{7}$$

with the jump condition

$$\left[\frac{f^2}{N^2}\psi_z\right]\Bigg|_{z=-\epsilon}^{z=\epsilon} = -2\Sigma, \quad \text{as} \quad \epsilon \to 0. \tag{8}$$

This implies that SQG corresponds to a fluid with zero potential vorticity everywhere within a fluid except on the surface $z = 0$. At that layer there is a potential vorticity sheet of strength Σ (with dimensions of velocity) that also corresponds to a potential temperature jump of

$$[\theta] = -\frac{2N^2}{f}\,\Sigma. \tag{9}$$

The horizontal velocity is continuous across this jump.

Taking the horizontal Fourier transform of ψ, and using the solution to Laplace's equation (7) we find a general solution

$$\psi(x,y,z) = \int\int e^{i(kx+ly)} F(k,l) e^{-\frac{N}{f}\kappa|z|}\, dkdl \tag{10}$$

where $\kappa = \sqrt{k^2+l^2}$. Let the Fourier transform of Σ be $\tilde{\Sigma}$, therefore

$$\begin{aligned} -2\tilde{\Sigma} &= \frac{f^2}{N^2}\left[-\frac{N}{f}\kappa e^{-\frac{N}{f}\kappa|0^+|} - \frac{N}{f}\kappa e^{-\frac{N}{f}\kappa|0^-|}\right]\, F(k,l) \\ F(k,l) &= \frac{N}{f}\,\frac{\tilde{\Sigma}}{\kappa}. \end{aligned} \tag{11}$$

Therefore we have the stream function throughout the domain

$$\psi(x,y,z) = \frac{N}{f}\int\int \tilde{\Sigma}\, e^{i(kx+ly)}\,\frac{e^{-\frac{N}{f}\kappa|z|}}{\kappa}\, dkdl \tag{12}$$

Now if we substitute (6) into (2) we find the surface quasi-geostrophic equation

$$\Sigma_t + U\Sigma_x + V\Sigma_y = 0 \tag{13}$$

where

$$U = -\Psi_y, \quad V = \Psi_x \tag{14}$$

and

$$\Psi(x,y) = \frac{N}{f}\int\int \tilde{\Sigma}\, e^{i(kx+ly)}\,\frac{1}{\kappa}\, dkdl. \tag{15}$$

We see that SQG is the advection of a temperature jump at the interface. Written more concisely, SQG is

$$\Sigma_t + \vec{U}\cdot\vec{\nabla}\Sigma = 0 \tag{16}$$

where

$$\vec{U} = \frac{N}{f}\,\nabla^{\perp}\,(-\Delta)^{-\frac{1}{2}}\,\Sigma. \tag{17}$$

2.1 The no-penetration boundary condition as it relates to SQG

SQG gives us an extra insight into the boundary condition (5). A rigid surface behaves as though it carries a temperature jump, or equivalently, a potential vorticity sheet. Consider a semi-infinite layer in $z > 0$ with no-penetration lower boundary condition and no potential vorticity in the interior. Then then the stream function satisfies

$$\psi_{xx} + \psi_{yy} + \frac{f^2}{N^2}\psi_{zz} = 0 \quad \text{in} \quad z > 0 \tag{18}$$

and

$$\psi_z = \frac{\theta}{f}, \quad \text{on} \quad z = 0. \tag{19}$$

So the stream function satisfies

$$\psi(x, y, z) = -\frac{1}{N} \int\int \tilde{\theta}\, e^{i(kx+ly)}\, \frac{e^{-\frac{N}{f}\kappa|z|}}{\kappa}\, dkdl. \tag{20}$$

More concisely we can write

$$\theta_t + \vec{U} \cdot \vec{\nabla}\theta = 0 \tag{21}$$

where

$$\vec{U} = -\frac{1}{N}\, \nabla^{\perp}\, (-\Delta)^{-\frac{1}{2}}\, \theta \tag{22}$$

on $z = 0$. Therefore, SQG describes a rigid boundary if we replace Σ from SQG by the temperature

$$\Sigma = -\frac{f}{N^2}\, \theta. \tag{23}$$

2.2 Two PV Sheets

J.K. Hunter considered a second PV sheet at $z = h$. The dynamics of the two sheets is

$$\begin{aligned} \Sigma_t + \vec{U}_0 \cdot \vec{\nabla}\Sigma &= 0 \\ \Gamma_t + \vec{U}_h \cdot \vec{\nabla}\Sigma &= 0. \end{aligned} \tag{24}$$

where

$$\psi(x, y, z) = \frac{N}{f} \int\int \left[\tilde{\Sigma}\, \frac{e^{-\frac{N}{f}\kappa|z|}}{\kappa} + \tilde{\Gamma}\, \frac{e^{-\frac{N}{f}\kappa|z-h|}}{\kappa}\right] e^{i(kx+ly)}\, dkdl \tag{25}$$

and

$$\begin{aligned} \vec{U}_0 &= \nabla^{\perp}\psi, \quad \text{on} \quad z = 0 \\ \vec{U}_h &= \nabla^{\perp}\psi, \quad \text{on} \quad z = h. \end{aligned} \tag{26}$$

Let $\Psi_0 = \psi(x, y, 0)$ and $\Psi_h = \psi(x, y, h)$, then

$$\begin{aligned} \Psi_0 &= \frac{N}{f}\, (-\Delta)^{-\frac{1}{2}} \left[\Sigma + e^{-\frac{N}{f}\, h|\vec{\nabla}|}\Gamma\right] \\ \Psi_h &= \frac{N}{f}\, (-\Delta)^{-\frac{1}{2}} \left[e^{-\frac{N}{f}\, h|\vec{\nabla}|}\Sigma + \Gamma\right]. \end{aligned} \tag{27}$$

3 Baroclinic Instability

3.1 Equilibrium

It is easier to describe the equilibrium using the full QG system. Consider two PV sheets at $z = 0, h$. Below $z = 0$ let the flow velocity be $\vec{u} = 0$ and the potential temperature be $\theta = 0$.

Let the PV sheet at $z = 0$ have strength ay and the one at $z = 0$ have strength βy, therefore

$$q = -\frac{f^2}{N^2}\left[\frac{U_*}{h} y\delta(z) + \frac{U_\infty}{h} y\delta(z-h)\right]. \tag{28}$$

We can see that the stream function for this potential vorticity distribution is

$$\psi = \begin{cases} 0 & z < 0 \\ -\frac{U_*}{h} yz & 0 < z < h \\ -U_* y - (U_* + U_\infty)\, y\left(\frac{z}{h} - 1\right) & z > h \end{cases} \tag{29}$$

This corresponds to a velocity field which depends on height alone

$$U(z) = \begin{cases} 0 & z < 0 \\ U_* \frac{z}{h} & 0 < z < h \\ U_* + (U_* + U_\infty)\left(\frac{z}{h} - 1\right) & z > h \end{cases} \tag{30}$$

and a temperature field which is constant in layers, but depends on y

$$\Theta(z) = -\frac{fy}{h} \begin{cases} 0 & z < 0 \\ U_* & 0 < z < h \\ U_* + U_\infty & z > h \end{cases} \tag{31}$$

In terms of the equilibrium PV densities on the surfaces we have

$$\bar{\Sigma} = \frac{U_* f^2 y}{2N^2 h}, \quad \bar{\Gamma} = \frac{U_\infty f^2 y}{2N^2 h}. \tag{32}$$

3.2 Linear Perturbation around the equilibrium

We will proceed with a standard perturbation around the equilibrium shear flow

$$\begin{aligned} \Sigma &= \bar{\Sigma} + \Sigma' \\ \Gamma &= \bar{\Gamma} + \Gamma' \\ \vec{U}_0 &= (U(0) + U_0')\,\hat{\imath} + V_0'\,\hat{\jmath} = U_0'\hat{\imath} + V_0'\,\hat{\jmath} \\ \vec{U}_h &= (U(h) + U_h')\,\hat{\imath} + V_h'\,\hat{\jmath} = (U_* + U_h')\,\hat{\imath} + V_h'\,\hat{\jmath} \end{aligned} \tag{33}$$

and linearization of the resulting equations

$$\begin{aligned} \Sigma_t' + V_0'\,\bar{\Sigma}_y &= 0 \\ \Gamma_t' + V_h'\,\bar{\Gamma}_y + U_*\,\Gamma_x' &= 0 \end{aligned} \tag{34}$$

with the perturbation y-velocities given by

$$
\begin{aligned}
V_0' &= \frac{N}{f}\frac{\partial}{\partial x}\left(-\Delta\right)^{-\frac{1}{2}}\left[\Sigma' + e^{-\frac{N}{f}h|\vec{\nabla}|}\Gamma'\right] \\
V_h' &= \frac{N}{f}\frac{\partial}{\partial x}\left(-\Delta\right)^{-\frac{1}{2}}\left[e^{-\frac{N}{f}h|\vec{\nabla}|}\Sigma' + \Gamma'\right].
\end{aligned}
\tag{35}
$$

Considering normal modes, all fields are proportional to

$$e^{ik(x-ct)+ily} \tag{36}$$

where $c(k,l) = \omega(k,l)/k$ is the complex valued wave speed in the x direction. We need to short hand the notation a bit otherwise it will get cumbersome. Define the coupling factor

$$\alpha = e^{-\frac{N}{f}h\kappa}. \tag{37}$$

The eigenvalue problem for c becomes

$$
\begin{aligned}
-c\,\Sigma' + \frac{N}{f}\frac{1}{\kappa}\left[\Sigma' + \alpha\Gamma'\right]\frac{U_* f^2}{2N^2 h} &= 0 \\
\left(U_* - c\right)\Gamma' + \frac{N}{f}\frac{1}{\kappa}\left[\alpha\Sigma' + \Gamma'\right]\frac{U_\infty f^2}{2N^2 h} &= 0,
\end{aligned}
\tag{38}
$$

It is now appropriate to non-dimensionalize. Define the scaled wavenumber

$$\mu = \frac{2N\kappa h}{f} \tag{39}$$

so that

$$\alpha = e^{\frac{\mu}{2}}. \tag{40}$$

which simplifies to

$$
\begin{aligned}
-2N\kappa hc\,\Sigma' + \left[\Sigma' + \alpha\Gamma'\right]U_* f &= 0 \\
2N\kappa h\left(U_* - c\right)\Gamma' + \left[\alpha\Sigma' + \Gamma'\right]U_\infty f &= 0,
\end{aligned}
\tag{41}
$$

and finally

$$
\begin{aligned}
\left(U_* f - 2N\kappa hc\right)\,\Sigma' + \alpha U_* f\,\Gamma' &= 0 \\
\alpha U_\infty f\,\Sigma' + \left[U_\infty f + 2N\kappa h\left(U_* - c\right)\right]\Gamma' &= 0.
\end{aligned}
\tag{42}
$$

Taking the determinant of this pair of equations yields

$$\left(U_* f - 2N\kappa hc\right)\left[U_\infty f + 2N\kappa h\left(U_* - c\right)\right] = \alpha^2 f^2 U_* U_\infty \tag{43}$$

Now, let

$$\gamma = \frac{f}{2N\kappa h}. \tag{44}$$

The eigenvalue problem is

$$\begin{aligned} \left[c - U_* - \gamma U_\infty\right]\left[c - \gamma U_*\right] - \alpha^2\gamma^2 U_* U_\infty = 0 \\ c^2 - \left(U_* + \gamma\left(U_* + U_\infty\right)\right)c + \gamma U_*^2 + \left(1 - \alpha^2\right)\gamma^2 U_* U_\infty = 0 \end{aligned} \tag{45}$$

and the wave speed is

$$\begin{aligned} c &= \frac{U_* + \gamma\left(U_* + U_\infty\right)}{2} \pm \frac{\sqrt{\left(U_* + \gamma\left(U_* + U_\infty\right)\right)^2 - 4\gamma U_*^2 - 4\left(1 - \alpha^2\right)\gamma^2 U_* U_\infty}}{2} \\ &= \frac{U_* + \gamma\left(U_* + U_\infty\right)}{2} \pm \frac{\sqrt{\left(U_* - \gamma\left(U_* + U_\infty\right)\right)^2 + 4\gamma U_* U_\infty\left[1 - \gamma\left(1 - \alpha^2\right)\right]}}{2} \\ &= \frac{U_* + \gamma\left(U_* + U_\infty\right)}{2} \pm \frac{U_*\sqrt{\left(1 - \gamma\left(1 + r\right)\right)^2 + 4\gamma r\left[1 - \gamma\left(1 - \alpha^2\right)\right]}}{2} \end{aligned} \tag{46}$$

where

$$r = \frac{U_\infty}{U_0} \tag{47}$$

is the ratio of the two velocity gradients - it can take any value. This is the simplest form of the eigenvalue. Notice that

$$\gamma = \frac{f}{2N\kappa h} \tag{48}$$

is a non-dimensional wavelength and

$$\alpha^2 = e^{-\frac{2N\kappa h}{f}} = e^{-\frac{1}{\gamma}} \tag{49}$$

is the exponential of a reciprocal of that wavelength (or the exponential of the wavenumber).

I see now that I should have used the reciprocal of γ, which is a scaled wavenumber, throughout the computation. Let

$$\mu = \gamma^{-1} = \frac{2N\kappa h}{f}, \quad \alpha^2 = e^{-\mu} \tag{50}$$

so that the eigenvalue is

$$c = \frac{U_*}{2}\left\{1 + \frac{1 + r}{\mu} \pm \sqrt{\left(1 - \frac{1 + r}{\mu}\right)^2 + \frac{4r\left(e^{-\mu} - \left(1 - \mu\right)\right)}{\mu^2}}\right\} \tag{51}$$

where $-\infty < r < \infty$ and $\mu > 0$.

The term outside the square root is always real. Inside the square root could be positive or negative - corresponding to growth or decay.

The case $r = 0$ corresponds to a single PV sheet with eigenvalues $c = U_*$ and $c = U_*/\mu$. The first root is spurious because there is no upper surface. The second root corresponds to constant frequency waves on the lower boundary.

Since $e^{-\mu} - (1 - \mu)$ is always positive, then a necessary condition for instability is that $r < 0$. The simple case $r = -1$ corresponds to no shear above the second PV sheet - i.e. the velocity is constant there.

In figure 1 is contoured the growth rate, $c_i\,\mu$ as a function of μ and r. It is clear that $r=-1$ has the strongest growth rates - and they occur around $\mu=1.5$. For all values of r, baroclinic instability is scale selective. For large negative values of r - which correspond to a strong counter shear above the PV sheet at $z=h$ - the growth rates decrease significantly and the unstable scales get large. The band of unstable modes also gets narrower at larger growth rates.

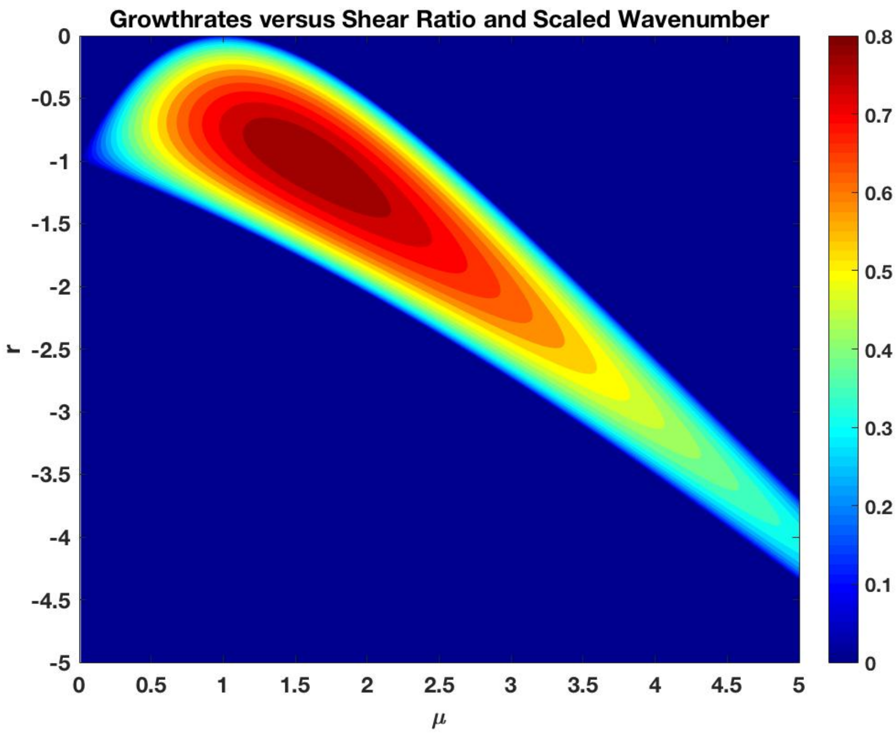

Figure 1: Growth rates versus scaled wavenumber μ and velocity gradient ratio, r. Positive values of r mean that the shear is greater above $z=h$ than below. $r=0$ corresponds to a constant shear - effectively no second SQG sheet at $z=h$. $r=-1$ corresponds to a PV sheet above which the velocity is constant. $r<-1$ has the velocity

Finally, we note that $\mu=2$ is a good approximation for the most unstable wavelength, for a wide range of velocity parameters. We find

$$\kappa = \frac{1}{h}\,\frac{f}{N}. \tag{52}$$

For typical atmospheric parameters, $f = 2\Omega\sin(\lambda) \approx \sqrt{2}\times 2\pi\times(86400\mathrm{s})^{-1} \approx 1.03\times 10^{-4}\,\mathrm{s}^{-1}$. The Brunt-Vaisala frequency is $N \approx 10^{-2}\,\mathrm{s}^{-1}$. Therefore $f/N \approx 10^{-2}$, and the baroclinic instability picks out scales which are about 100 times the depth of the velocity gradient layer - about 1500 km.