On the equations of a dog pursuing a duck

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1 Derivation of the equations

We consider the problem of a duck swimming in a circle in a pond. The duck's trajectory is given by

$$\vec{x}_{\text{Duck}}(t) = a\cos(\omega t)\hat{i} + a\sin(\omega t)\hat{j} \tag{1}$$

and we will define

$$\omega t = \theta. \tag{2}$$

Its speed is constant and equal to

$$\left|\frac{d}{dt}\vec{x}_{\text{Duck}}\right| = a\omega. \tag{3}$$

A dog is located at the point

$$\vec{x}_{\text{Dog}}(t) = x(t)\hat{i} + y(t)\hat{j} \tag{4}$$

The vector pointing from the dog to the duck is

$$\vec{R}(t) = \vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t), \tag{5}$$

its magnitude is

$$R(t) = |\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)| \tag{6}$$

and the unit vector in that direction is

$$\frac{\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)}{|\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)|}. \tag{7}$$

The dynamics of the dog's motion are simple. It swims at a constant speed, s, in the direction of the duck. Therefore, it is always swimming in the unit vector direction pointing from the dog to the duck. We can write this system of equations as

$$\frac{d\,\vec{x}_{\text{Dog}}}{dt} = s\frac{\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)}{|\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)|} \tag{8}$$

Figure 1 shows the pursuit diagram, which is the same as in Strogatz.

Equations (8) are a 2nd order system of non-autonomous ODEs. We will find a simpler set of equations if we express this system in terms of the distance between the duck and the

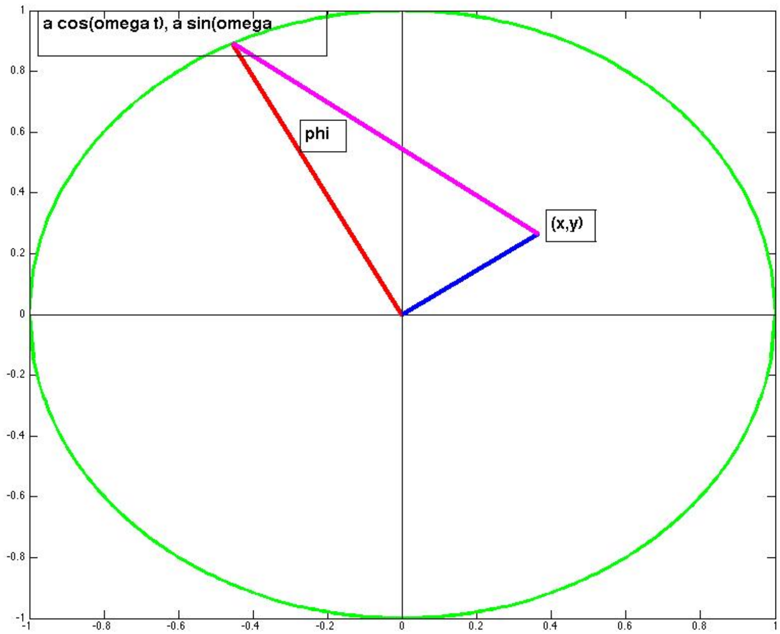

Figure 1: The geometry of the pursuit problem.

dog, R and the angle ϕ between the vector connecting the duck and the dog, and the vector between the duck and the origin,

$$\cos(\phi) = \frac{[\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)] \cdot \vec{x}_{\text{Duck}}(t)}{|\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)| \; |\vec{x}_{\text{Duck}}(t)|} = \frac{\vec{R} \cdot \vec{x}_{\text{Duck}}(t)}{R\, a} \tag{9}$$

We need to define the coordinate transformation between (x, y) and (R, ϕ) Notice that

$$\begin{aligned} \vec{x}_{\text{Dog}}(t) &= \vec{x}_{\text{Duck}}(t) - \vec{R} \\ x\hat{i} + y\hat{j} &= a\left(\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}\right) - R\left(\cos(\alpha)\hat{i} + \sin(\alpha)\hat{j}\right). \end{aligned} \tag{10}$$

Now we have to express the angle α, which is the angle the magenta line makes with the horizontal, in terms of θ, which is the angle the red line makes with the horizontal, and ϕ, which is the angle between the red and the magenta lines. If we were to extend the magenta line down to the horizontal axis, then we see that there is a triangle which is made by the extended magenta line, the red line, and the horizontal black line. The interior angles of this triangle are $\pi - \alpha, \theta$ and ϕ, therefore

$$(\pi - \alpha) + \theta + \phi = \pi \quad \Longrightarrow \quad \alpha = \theta + \phi \tag{11}$$

And we find

$$x\hat{i} + y\hat{j} = a\left(\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j}\right) - R\left(\cos(\theta + \phi)\,\hat{i} + \sin(\theta + \phi)\,\hat{j}\right) \tag{12}$$

or

$$x = a\cos(\theta) - R\cos(\theta + \phi)\,, \quad y = a\sin(\theta) - R\sin(\theta + \phi) \tag{13}$$

where I have replaced $\omega t = \theta$. Another fact to point out is that we have determined

$$\vec{R} = R\left(\cos(\theta + \phi)\,\hat{\imath} + \sin(\theta + \phi)\,\hat{\jmath}\right) \tag{14}$$

The next thing to do is to express the equations in terms of R, ϕ. Take (8) and subtract $\frac{d\,\vec{x}_{\text{Duck}}}{dt}$ from both sides of the equation,

$$\begin{aligned} -\frac{d}{dt}\left[\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)\right] &= s\frac{\vec{x}_{\text{Duck}}(t) - \vec{x}_{\text{Dog}}(t)}{R} - a\omega\left[-\sin(\omega t)\hat{\imath} + \cos(\omega t)\hat{\jmath}\right] \\ -\frac{d\vec{R}}{dt} &= s\frac{\vec{R}}{R} - a\omega\left[-\sin(\omega t)\hat{\imath} + \cos(\omega t)\hat{\jmath}\right] \end{aligned} \tag{15}$$

and take the dot product with $\vec{R}$, recognize $\vec{R}\cdot\vec{R} = R^2$, and replace $\omega t = \theta$ to find

$$\begin{aligned} -\vec{R}\cdot\frac{d\vec{R}}{dt} &= s\,\frac{\vec{R}\cdot\vec{R}}{R} - a\omega\vec{R}\cdot\left[-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right] \\ -\frac{1}{2}\frac{d}{dt}R^2 &= sR - a\omega R\left[\sin(\theta + \phi)\cos(\theta) - \cos(\theta + \phi)\sin(\theta)\right] \\ -R\frac{dR}{dt} &= sR - a\omega R\sin(\theta + \phi - \theta) \end{aligned} \tag{16}$$

therefore the distance between the duck and the dog is governed by

$$\frac{dR}{dt} = a\omega\sin(\phi) - s. \tag{17}$$

To get the equation for the evolution of the angle, ϕ, take the definition of ϕ

$$
\begin{aligned}
\cos(\phi) &= \frac{\vec{R}\cdot\vec{x}_{\text{Duck}}}{Ra} \\
Ra\cos(\phi) &= \vec{R}\cdot\vec{x}_{\text{Duck}} \\
a\frac{dR}{dt}\cos(\phi) - aR\sin(\phi)\frac{d\phi}{dt} &= \frac{d\vec{R}}{dt}\cdot\vec{x}_{\text{Duck}} + \vec{R}\cdot\frac{d\vec{x}_{\text{Duck}}}{dt} \\
a\left[a\omega\sin(\phi) - s\right]\cos(\phi) - aR\sin(\phi)\frac{d\phi}{dt} &= \left[-s\frac{\vec{R}}{R} + a\omega\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right)\right]\cdot a\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) \\
&\quad + \vec{R}\cdot a\omega\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) \\
a\left[a\omega\sin(\phi) - s\right]\cos(\phi) - aR\sin(\phi)\frac{d\phi}{dt} &= -as\frac{\vec{R}}{R}\cdot\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) + a\omega\vec{R}\cdot\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) \\
\left[a\omega\sin(\phi) - s\right]\cos(\phi) - R\sin(\phi)\frac{d\phi}{dt} &= -s\frac{\vec{R}}{R}\cdot\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) + \omega\vec{R}\cdot\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) \\
\left[a\omega\sin(\phi) - s\right]\cos(\phi) - R\sin(\phi)\frac{d\phi}{dt} &= -s\left(\cos\left(\theta+\phi\right)\hat{\imath} + \sin\left(\theta+\phi\right)\hat{\jmath}\right)\cdot\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right) \\
&\quad + \omega R\left(\cos\left(\theta+\phi\right)\hat{\imath} + \sin\left(\theta+\phi\right)\hat{\jmath}\right)\cdot\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right) \\
\left[a\omega\sin(\phi) - s\right]\cos(\phi) - R\sin(\phi)\frac{d\phi}{dt} &= -s\left[\cos\left(\theta+\phi\right)\cos(\theta) + \sin\left(\theta+\phi\right)\sin(\theta)\right] \\
&\quad + \omega R\left[-\cos\left(\theta+\phi\right)\sin(\theta) + \sin\left(\theta+\phi\right)\cos(\theta)\right] \\
\left[a\omega\sin(\phi) - s\right]\cos(\phi) - R\sin(\phi)\frac{d\phi}{dt} &= -s\cos(\theta+\phi-\theta) + \omega R\sin(\theta+\phi-\theta) \\
a\omega\sin(\phi)\cos(\phi) - s\cos(\phi) - R\sin(\phi)\frac{d\phi}{dt} &= -s\cos(\phi) + \omega R\sin(\phi) \\
a\omega\sin(\phi)\cos(\phi) - R\sin(\phi)\frac{d\phi}{dt} &= \omega R\sin(\phi) \\
a\omega\cos(\phi) - R\frac{d\phi}{dt} &= \omega R
\end{aligned}
\tag{18}
$$

and finally

$$\frac{d\phi}{dt} = \frac{a\omega\cos(\phi)}{R} - \omega. \tag{19}$$

Notice that these are now *autonomous equations*, meaning that θ does not appear explicitly on the right hand sides. Now we are going to recast these equations (17) and (19) in terms of the independent variable

$$\theta = \omega t \tag{20}$$

and scale the distance in units of the radius of the circle

$$r = \frac{R}{a} \tag{21}$$

and define the ratio of the dog's speed to the duck's speed as

$$k = \frac{s}{a\omega} \tag{22}$$

to find the equations

$$\begin{aligned} \frac{dr}{d\theta} &= \sin(\phi) - k \\ \frac{d\phi}{d\theta} &= \frac{\cos(\phi)}{r} - 1. \end{aligned} \tag{23}$$

These are the pursuit equations which depend on one parameter, $k \geq 0$.

2 Equilibrium

There are no equilibria if $k > 1$. If $0 \leq k \leq 1$ we have only one equilibrium

$$\sin(\phi_*) = k, \quad r_* = \cos(\phi_*) = \sqrt{1 - \sin^2(\phi_*)} = \sqrt{1-k^2} \leq 1 \tag{24}$$

As a function of r, ϕ the equilibria are shown in figure 2. Visualized on the circle, the curve

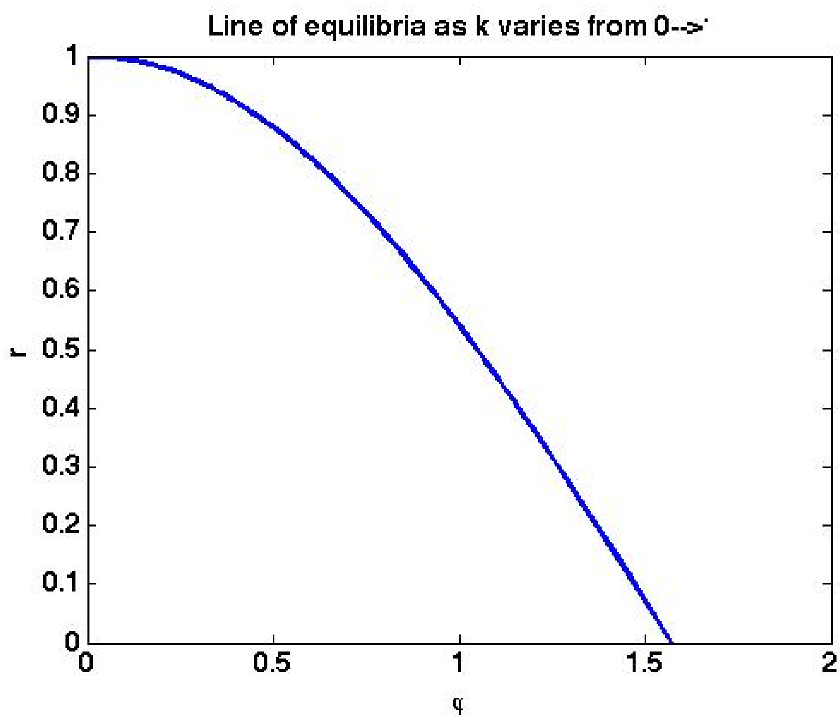

Figure 2: The equilibria all lie along the blue curve in the (ϕ, r) plane. As k varies from zero to one, r varies from one to zero.

of equilibria are shown in figure 3.

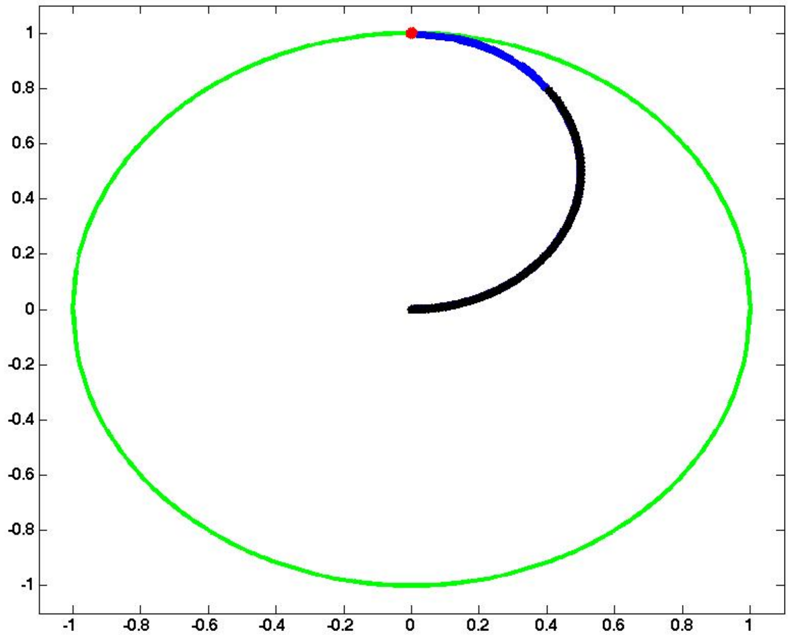

Figure 3: The duck is located at the red dot, the circle is in green. In the pursuit problem, the duck is rotating around the circle at constant angular velocity - so we are looking at the circle in the frame of reference of the duck. As k varies, the equilibria all lie along the blue or black curve. For $k = 0$ the equilibrium is at the center of the circle. For $k = 1$ the equilibrium is on the circle -i.e. on the duck. The black curve indicates the equilibria which are spiral stable, while the blue curve indicates equilibria which are stable nodes.

3 Stability

The stability of the equilibrium is governed by the Jacobian matrix

$$\begin{bmatrix} 0 & \cos(\phi_*) \\ -\frac{\cos(\phi_*)}{r_*^2} & -\frac{\sin(\phi_*)}{r_*} \end{bmatrix} \tag{25}$$

whose characteristic polynomial is

$$\lambda^2 + \lambda\frac{\sin(\phi_*)}{r_*} + \frac{\cos^2(\phi_*)}{r_*^2} = 0 \tag{26}$$

so that

$$
\begin{aligned}
\lambda &= -\frac{\sin(\phi_*)}{2r_*} \pm \frac{1}{2}\sqrt{\frac{\sin^2(\phi_*)}{r_*^2} - 4\frac{\cos^2(\phi_*)}{r_*^2}} \\
&= -\frac{1}{2r_*}\left[\sin(\phi_*) \mp \sqrt{\sin^2(\phi_*) - 4 + 4\sin^2(\phi_*)}\right] \\
&= -\frac{1}{2\sqrt{1-k^2}}\left[k \pm \sqrt{5k^2-4}\right]
\end{aligned}
\tag{27}
$$

When $k < \sqrt{\frac{4}{5}}$, i.e. $r_* \geq \sqrt{\frac{1}{5}}$, the eigenvalues are complex with negative real part. Therefore the equilibrium is a stable spiral.

When $\sqrt{\frac{4}{5}} < k < 1$ the roots are real and both are negative. Therefore the equilibrium is a stable node.