On the period of the Duffing Oscillator

J.A. Biello

March 16, 2018

Consider the oscillator with a quadratic potential energy (cubic equation)

$$\ddot{x} + \omega^2 x + \alpha x^3 = 0 \tag{1}$$

with initial data $\dot{x}(0) = 0$ and $x(0) = x_0$. Now rescale time by ω and x by x_0 so that

$$\ddot{x} + x + \epsilon x^3 = 0. \tag{2}$$

with $\dot{x}(0) = 0$ and $x(0) = 1$. Here we have

$$\epsilon = \frac{\alpha x_0^2}{\omega^2}, \tag{3}$$

and I have made the standard reuse of the original variables. Constructing the energy equation by multiplying by $\dot{x}$, we find

$$(\dot{x})^2 + x^2 + \epsilon \frac{x^4}{2} = 1 + \frac{\epsilon}{2}. \tag{4}$$

The energy curve is a closed ellipse-like shape in $(x, \dot{x})$ phase space. Separating variables we can write

$$\begin{aligned} (\dot{x})^2 + x^2 + \epsilon \frac{x^4}{2} &= 1 + \frac{\epsilon}{2} \\ (\dot{x})^2 &= 1 + \frac{\epsilon}{2} - \left(x^2 + \epsilon \frac{x^4}{2} \right) \\ \frac{dx}{\pm\sqrt{1 + \frac{\epsilon}{2} - \left(x^2 + \epsilon \frac{x^4}{2} \right)}} &= dt. \end{aligned} \tag{5}$$

Integrating this expression over one full circuit of the closed curve in the $x, \dot{x}$ phase space would yield the period of the orbit, T, on the right hand side. Instead, we can integrate from $x = 0...1$ to get the quarter period

$$\int_0^1 \frac{dx}{\sqrt{1 + \frac{\epsilon}{2} - \left(x^2 + \epsilon \frac{x^4}{2} \right)}} = \frac{T}{4} \tag{6}$$

The only issue now is to evaluate this expression. There is a tricky substitution that I found that clears things up. Let

$$u = \sqrt{\frac{x^2 + \frac{\epsilon x^4}{2}}{1 + \frac{\epsilon}{2}}} = x\,\sqrt{\frac{1 + \frac{\epsilon x^2}{2}}{1 + \frac{\epsilon}{2}}} \tag{7}$$

so that the denominator of the integral is

$$\sqrt{1 + \frac{\epsilon}{2} - \left(x^2 + \epsilon \frac{x^4}{2}\right)} = \sqrt{1 + \frac{\epsilon}{2}}\,\sqrt{1 - u^2}. \tag{8}$$

When $x = 0, u = 0$, when $x = 1, u = 1$ and

$$\begin{aligned} 2udu\left(1 + \frac{\epsilon}{2}\right) &= \left[2x + 2\epsilon x^3\right]\,dx \\ \frac{u\,\left(1 + \frac{\epsilon}{2}\right)}{x\left(1 + \epsilon x^2\right)} &= dx \\ dx &= \frac{\sqrt{1 + \frac{\epsilon x^2}{2}}}{1 + \epsilon x^2}\,\sqrt{1 + \frac{\epsilon}{2}} \end{aligned} \tag{9}$$

Substituting these expressions into the integral for the period we have

$$T = 4\int_0^1 \frac{du}{1 - u^2}\,\frac{\sqrt{1 + \frac{\epsilon x(u)^2}{2}}}{1 + \epsilon x(u)^2}, \tag{10}$$

where we write $x(u)$ because in order to solve this integral we have to express x in terms of u. The transformation above can be written

$$\begin{aligned} \frac{\epsilon}{2}\,x^4 + x^2 - \left(1 + \frac{\epsilon}{2}\right)\,u^2 &= 0 \\ x^2 &= \frac{-1 \pm \sqrt{1 + 2\epsilon\,u^2\,\left(1 + \frac{\epsilon}{2}\right)}}{\epsilon}. \end{aligned} \tag{11}$$

Taking the positive square root and then expanding in ϵ we find

$$\begin{aligned} \epsilon\,x^2 &= \sqrt{1 + 2\epsilon\,u^2\,\left(1 + \frac{\epsilon}{2}\right)}\; - 1 \\ &\approx \epsilon u^2\left(1 + \frac{\epsilon}{2}\right) - \frac{1}{8}\,\left[2\epsilon\,u^2\,\left(1 + \frac{\epsilon}{2}\right)\right]^2 + ... \\ &\approx \epsilon\,u^2\; + \frac{\epsilon^2}{2}\left(u^2 - u^4\right) + ... \end{aligned} \tag{12}$$

So the term in the integral that must be expanded in ϵ is

$$\frac{\sqrt{1 + \frac{\epsilon x(u)^2}{2}}}{1 + \epsilon x(u)^2} = \frac{\sqrt{1 + \frac{1}{2}\left[\epsilon\,u^2\; + \frac{\epsilon^2}{2}\left(u^2 - u^4\right) + ...\right]}}{1 + \epsilon\,u^2\; + \frac{\epsilon^2}{2}\left(u^2 - u^4\right) + ...} \tag{13}$$