The Van der Pol Oscillator in the large and small damping limits

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The Van der Pol oscillator is a nonlinearly damped oscillator whose nondimensionalized ODE expression is

$$\ddot{x} + \mu\left(x^2 - 1\right)\,\dot{x} + x = 0. \tag{1}$$

It is (apparently) an example of a more general Lienard system where the damping function is a more general (albeit with some restrictions) function of x.

1 The large damping limit, $\mu \gg 1$

The standard transformation

$$\mu y = \dot{x} + \mu\left[\frac{x^3}{3} - x\right] \tag{2}$$

casts the problem as a system of two equations

$$\begin{aligned} \dot{x} &= \mu\left[y - F(x)\right] \\ \dot{y} &= -\frac{x}{\mu} \end{aligned} \tag{3}$$

where

$$F(x) = \frac{x^3}{3} - x, \quad \frac{dF}{dx} = x^2 - 1. \tag{4}$$

The system in (3) is slow in y and fast in x, unless $y \approx F(x)$. We want to be able to "see" the dynamics, so we want to have a time variable which captures the main dynamics encompassed by this system. Therefore, define a new time variable

$$\tau \equiv \frac{t}{\mu}. \tag{5}$$

This means that when t is equal to one unit, τ is very small. In this sense, τ is a slow time variable, capturing the slower dynamics which are sitting in the $\dot{y}$ equation. In the new time variable

$$\begin{aligned} \frac{dx}{d\tau} &= \mu^2\left[y - F(x)\right] \\ \frac{dy}{d\tau} &= -x \end{aligned} \tag{6}$$

1.1 Asymptotic expansion

Let us consider an asymptotic expansion - essentially an infinite series expansion in μ - which will help us see a good approximation for these equations. We let

$$\begin{aligned} x &= x_0 + \frac{1}{\mu^2}\, x_1 + ... \\ y &= y_0 + \frac{1}{\mu^2}\, y_1 + ... \end{aligned} \tag{7}$$

In principle the approximation can be continued for arbitrarily high terms in this series. In practice, we only care about the first term or two in the the limit of large μ. Upon substituting this expansion into the equations we find

$$\begin{aligned} \frac{dx_0}{d\tau} + \frac{1}{\mu^2}\frac{dx_1}{d\tau} + ... &= \mu^2 \left[y_0 + \frac{1}{\mu^2} y_1 + ... - F\left(x_0 + \frac{1}{\mu^2} x_1 + ... \right) \right] \\ &= \mu^2 \left[y_0 + \frac{1}{\mu^2} y_1 + ... - F\left(x_0\right) - \frac{x_1}{\mu^2}\frac{dF(x_0)}{dx} + ... \right] \\ \frac{dy_0}{d\tau} + \frac{1}{\mu^2}\frac{dy_1}{d\tau} + ... &= -x_0 - \frac{1}{\mu^2} x_1 + ... \end{aligned} \tag{8}$$

where I have expanded $F(x)$ in a Taylor series around x_0 in the first equation.

The idea of an asymptotic theory is that we now collect like terms in powers of μ and equate them to each other. The largest term in the first equation is of power μ^2 and the largest term in the second equation is power of μ^0. Therefore we find

$$\begin{aligned} 0 &= y_0 - F(x_0) \\ \frac{dy_0}{d\tau} &= -x_0. \end{aligned} \tag{9}$$

The second largest term in the first equation is power μ^0 and the second largest term in the second equation is power μ^{-2}. Therefore we find

$$\begin{aligned} \frac{dx_0}{d\tau} &= y_1 - x_1\, \frac{dF(x_0)}{dx} \\ \frac{dy_1}{d\tau} &= -x_1. \end{aligned} \tag{10}$$

1.1.1 Lowest order solution

Looking at equation (9) we find the lowest order solution

$$y_0 = F(x_0) = \frac{x_0^3}{3} - x_0 \tag{11}$$

and, substituting this into the y_0 equation we find

$$\frac{dx_0}{d\tau} = -\frac{x_0}{x_0^2 - 1} \tag{12}$$

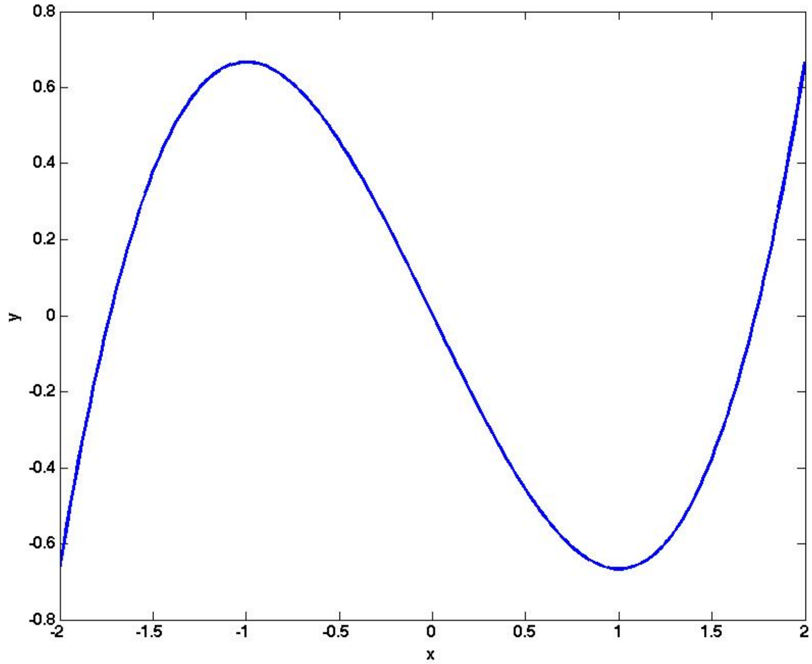

Figure 1: The cubic, $y_0 = F(x_0)$, for the lowest order balance of the Van der Pol oscillator.

The function $y_0 = F(x_0)$ is plotted in figure 1. So we conclude from this that, for any initial condition, trajectories move horizontally (i.e. at nearly constant y) until they reach $y = F(x)$. Above the graph they move to the right and below the graph they move to the left.

When on the graph, they move according to (12), which is a first order ODE. Equation (12) can be written as the gradient of a potential

$$\frac{dx_0}{d\tau} = -\frac{dV(x_0)}{dx_0} \tag{13}$$

where

$$V(x) = \frac{1}{2} \ln \left| x^2 - 1 \right| . \tag{14}$$

The potential is plotted in figure 2 and we can see that all trajectories "roll down the potential to the points $x_0 = \pm 1$. Notice that the potential becomes unbounded as $x \to \pm 1$, which is an indication that the approximation breaks down. If the trajectory actually went to $x = \pm 1$, then these would be fixed points of the system - which they are not. The points where the approximation breaks down are the extrema of the function $F(x)$. It is near these extrema that we have to reconsider the solution.

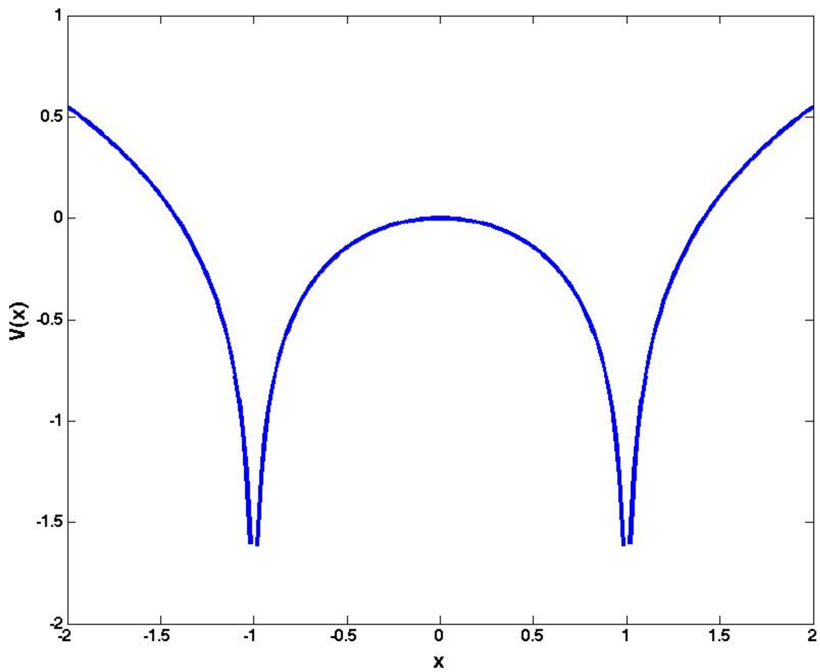

Figure 2: The potential for the first order equation in the Van der Pol oscillator.

1.2 Near $x = \pm 1$, $y = \mp\frac{2}{3}$

In this case we can simple return to the original form of the oscillator and realize there is no issue whatsoever

$$\ddot{x} + x = 0 \tag{15}$$

which is the simple harmonic oscillator. In terms of the variable τ is

$$\frac{d^2x}{d\tau^2} + \mu^2 x = 0. \tag{16}$$

So, the reason everything "blows up" near the extrema of the cubic is that the dynamics are happening faster than time variable, τ, can capture.

2 The small damping limit, $\mu \ll 1$

In this limit we can use the original form of the Van der Pol oscillator and simply write an asymptotic expansion

$$x = x_0 + \mu x_1 + ... \tag{17}$$

At the lowest order in μ we find

$$\ddot{x}_0 + x = 0 \tag{18}$$

and the next order gives

$$\ddot{x}_1 + x_1 = -\left(x_0^2 - 1\right)\dot{x}_0. \tag{19}$$

I have written the equation with x_0 terms on the right hand side because we think of these equations as follows. We solve the lowest order theory for $x_0(t)$ and then the next order theory looks like an inhomogeneous linear equation for $x_1(t)$.

The lowest order solution is

$$x_0(t) = R\cos(t+\phi) = \frac{Re^{i(t+\phi)} + Re^{-i(t+\phi)}}{2} \tag{20}$$

The inhomogeneous first order equation is

$$\begin{aligned}
\ddot{x}_1 + x_1 &= R\sin(t+\phi)\left(R^2\cos^2(t+\phi) - 1\right) \\
&= R\left[\frac{e^{i(t+\phi)} - e^{-i(t+\phi)}}{2i}\right]\left\{\frac{R^2}{4}\,\left[e^{2i(t+\phi)} + 2 + e^{-2i(t+\phi)}\right] - 1\right\} \\
&= e^{3i(t+\phi)}\,\frac{R^3}{8i} + e^{i(t+\phi)}\,\frac{R}{2i}\left\{\frac{2R^2}{4} - 1 - \frac{R^2}{4}\right\} - e^{-i(t+\phi)}\,\frac{R}{2i}\left\{\frac{2R^2}{4} - 1 - \frac{R^2}{4}\right\} - e^{-3i(t+\phi)}\,\frac{R^3}{8i} \\
&= \sin(t+\phi)\left\{\frac{R^3}{4} - R\right\} + \sin\left(3\left(t+\phi\right)\right)\,\frac{R^3}{4}.
\end{aligned} \tag{21}$$

The way you solve this equation is that you find, first, the homogenous solution

$$x = R_1\cos(t+\phi_1) \tag{22}$$

and then you find the particular solution for each of the inhomogeneous terms on the right hand side.

You can check that

$$x = -\sin\left(3\left(t+\phi\right)\right)\,\frac{R^3}{32} \tag{23}$$

solve the equation for the second inhomogeneous term. However, the first inhomogeneous term has the same oscillation frequency as the frequency of the inhomogeneous oscillator. So if we guess a solution $x \propto \sin(t+\phi)$ then we will find that it cannot be a solution with that inhomogeneity. Instead we guess a form

$$x = At\cos(t+\phi), \quad \dot{x} = A\cos(t+\phi) - At\sin(t+\phi), \quad \ddot{x} = -2A\sin(t+\phi) - At\cos(t+\phi) \tag{24}$$

which we substitute into the equation to find

$$-2A\sin(t+\phi) = \sin(t+\phi)\left\{\frac{R^3}{4} - R\right\} \tag{25}$$

meaning that

$$A = \frac{1}{2}\left[R - \frac{R^3}{4}\right] \tag{26}$$

If we right the whole solution down for x_1 we have

$$x_1 = R_1 \cos(t + \phi_1) + \frac{1}{2}\left[R - \frac{R^3}{4}\right] t \sin(t + \phi) - \sin\left(3\left(t + \phi\right)\right) \frac{R^3}{32}. \tag{27}$$

Notice that the first and last terms are always oscillating, so their magnitude never grows beyond a certain limit ($R^3/32$, for example). On the other hand the middle term is an oscillating function multiplied by t - so it keeps growing as time increases.

But this increase means that x_1 will, at some time, grow bigger than x_0 - and this means that x_0 does not make a good approximation of the solution anymore. This time is when

$$\mu t \approx 1 \tag{28}$$

which for small μ is a large time.

2.1 The method of multiple timescales.

One of the most powerful, and sometimes mind boggling, methods used in asymptotics is the method of multiple timescales. The idea is that we have already seen there are 2 time scales in the problem - a fast one associated with the oscillation and a slow one ($t \sim \mu^{-1}$) associated with the resonance. We try to capture the dynamics on the slow time scale by introducing a second variable describing time and then assuming the two time variables are independent. This is a heckuva stretch and you should not necessarily take my word for it that it works. But I will show you how it works and then you can try to concoct examples on your own. The idea is to let

$$\tau = \mu t \tag{29}$$

so that when t is large (a lot of time has transpired) then τ is about equal to one. Then, and this is the mind bender, we assume that we need to study the equations on both the t *and* the τ timescale. Therefore we replace the time derivative by

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \tau}. \tag{30}$$

With this replacement, the second derivative becomes

$$\frac{d^2}{dt^2} \rightarrow \frac{\partial^2}{\partial t^2} + 2\mu \frac{\partial^2}{\partial \tau \partial t} + \mu^2 \frac{\partial^2}{\partial \tau^2} \tag{31}$$

These new terms on the right side are small, but they have to be included at the higher orders of the theory. The equations become

$$\frac{\partial^2 x}{\partial t^2} + 2\mu \frac{\partial^2 x}{\partial \tau \partial t} + \mu^2 \frac{\partial^2 x}{\partial \tau^2} + \mu\left(x^2 - 1\right)\left[\frac{\partial x}{\partial t} + \mu \frac{\partial x}{\partial \tau}\right] + x = 0. \tag{32}$$

Again we substitute the asymptotic expansion and we find that the lowest order of the theory is unchanged

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \tag{33}$$

except for the partial derivative appearing in this version. The solution is similar to before except the terms that were parameters in the previous solution now become functions of the slow time variable

$$x_0(t,\tau) = R(\tau)\cos(t+\phi(\tau)). \tag{34}$$

In the first order correction only one new term appears

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = -2\frac{\partial^2 x_0}{\partial\tau\partial t} + \left(x_0^2 - 1\right)\frac{\partial x_0}{\partial t}. \tag{35}$$

The right hand side is exactly the same as we've already calculated, except for the first term, which is

$$-2\frac{\partial^2 x_0}{\partial\tau\partial t} = \frac{\partial}{\partial\tau}\left[2R(\tau)\sin(t+\phi(\tau))\right] = 2\frac{dR}{d\tau}\sin(t+\phi(\tau)) + 2R\frac{d\phi}{d\tau}\cos(t+\phi) \tag{36}$$

and the first order equation becomes

$$\ddot{x}_1 + x_1 = 2\frac{dR}{d\tau}\sin(t+\phi(\tau)) + 2R\frac{d\phi}{d\tau}\cos(t+\phi) + \sin(t+\phi)\left\{\frac{R^3}{4} - R\right\} + \sin\left(3\left(t+\phi\right)\right)\ \frac{R^3}{4}. \tag{37}$$

Finally, we recognize that all the terms on the right hand side that are resonant with the homogeneous solution will only cause the solution x_1 to increase with time. If we want to avoid this, we have to remove the resonant terms on the right hand side by equating them to one another - this is called the Fredholm alternative. Therefore

$$2\frac{dR}{d\tau} + \left[\frac{R^3}{4} - R\right] = 0 \tag{38}$$

and

$$2R\frac{d\phi}{d\tau} = 0. \tag{39}$$

These two equations say that the phase is constant, but that the radius of the orbit increases according to

$$\frac{dR}{d\tau} = \frac{R}{2}\left[1 - \frac{R^2}{4}\right]. \tag{40}$$

This is a simple one dimensional system that looks a lot like the pitchfork bifurcation. It has equilibria at

$$R = 0 \tag{41}$$

and

$$R = 2. \tag{42}$$

The second equilibrium is a circle of radius 2 - which is the limit cycle.

2.2 Finding the limit cycle using an energy argument

Multiplying the equation for the Van der Pol oscillator (1) by $\dot{x}$ we find

$$\frac{d}{dt}\left[\frac{\dot{x}^2 + x^2}{2}\right] = -\mu\left(x^2 - 1\right)\dot{x}^2. \tag{43}$$

The right hand side is the time derivative of the total energy of an undamped oscillator. In the absence of damping, $\mu = 0$, the energy would be conserved and the solution would be a circle in the $(x, \dot{x})$ phase plane.

The general solution to the oscillator ODE is

$$x = R\cos(t + \phi) \tag{44}$$

with period $T = 2\pi$. If we integrate the energy over one period we find

$$\Delta E = -\mu \int_0^T \left(x(t)^2 - 1\right)\ \dot{x}(t)^2\, dt. \tag{45}$$

A simple, qualitative argument for finding the limit cycle goes as follows. Can we find the solution to the undamped oscillator equation (i.e. find R and ϕ) such that, when it is subjected to damping, the energy is still conserved? In order to have it conserved, the right hand side of (45) must be zero. Substitute (44) into the right hand side of (45) to find

$$\begin{aligned}
0 &= \int_0^T \left(R^2\cos^2(t+\phi) - 1\right)\ R^2\ \sin^2(t+\phi)\, dt \\
&= R^2 \int_0^T \left(R^2\cos^2(t) - 1\right)\ \sin^2(t)\, dt \quad \text{since the integrand is periodic} \\
&= \int_0^{2\pi} \left(R^2\cos^2(t)\sin^2(t) - \sin^2(t)\right)\ dt \\
&= \int_0^{2\pi} \left(\frac{R^2\ \sin^2(2t)}{4} - \sin^2(t)\right)\ dt.
\end{aligned} \tag{46}$$

Now the average of $\sin^2(nt)$ from $[0, 2\pi]$ is $\frac{1}{2}$, therefore each of the two terms in the integral evaluate to $2\pi \cdot \frac{1}{2} = \pi$ and we find

$$0 = \frac{R^2}{4} - 1 \quad \Longrightarrow \quad R = 2, \tag{47}$$

which is the same result we got from the asymptotic method. This method of averaging essentially creates the periodic orbit by finding the orbit over which the average dissipation is equal to zero.